Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 4: proof by induction
to be completed by Friday February 6 2015

1) Let $P(n)$ be the assertion $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$. Prove that $P(n)$ is true for all $n \geq 1$.

2) i) Evaluate $\sum_{i=1}^{n} i \cdot i!$ for $n = 1, 2, 3, 4, 5$.

   ii) Guess an expression for $\sum_{i=1}^{n} i \cdot i!$.

   iii) Prove by induction that your formula works for all n.

3) Let $P(n)$ be the assertion that $8^n - 3^n$ is divisible by 5. Prove that $P(n)$ is true for all $n \geq 1$.

4) Let $P(n)$ be the assertion that $3^{2n} \equiv 1 \mod 8$. Prove that $P(n)$ is true for all $n \geq 1$.

5) i) Use the binomial theorem and induction to prove Fermat's Little Theorem in the form stated here: Fix a prime p. For any positive integer n, $n^p \equiv n \mod p$.

   ii) Deduce Fermat's Little Theorem in the form proved in class: $n^{p-1} \equiv 1 \mod p$ if $p \nmid n$. Why is the hypothesis needed for this version, but not in i)?

6) Let $S_n = \{a_1, a_2, \ldots, a_n\}$ be a set with n elements.

   i) List all subsets of S_1, S_2, S_3 and S_4. How many subsets does each one have? (Remember that the empty set and the set itself are both subsets of a given set.)

   ii) Find an organized way to list the subsets of S_5, given that you already have a list of the subsets of S_4. Do the same for S_{n+1}, assuming that you have a list of all subsets of S_n.

   iii) Prove by induction that for every positive integer n, S_n has 2^n subsets.