Math 1C03 Problem Sheet 2 Solutions

1) i) ii) done in class

iii) If $a \mid b$ then there is an integer $n \geq 1$ such that $b = na$ ($n \geq 1$ provided $b \neq 0$. If $b = 0$ the statement is false, as every number divides 0).

Then $|b| = |na| = |n||a| = n|a|$.

Thus $|b| \geq |a|$, as $n \geq 1$.

2) i) Consider $a = 7$, $b = 2$. Then $7 = 2 \times 3 + 1$, and $q = 3$, $r = 1$ are unique, if $0 \leq r < b = 2$.
Without the assumption on the size of r, we can write $7 = 2 \cdot 2 + 3$, $7 = 2 \cdot 1 + 5$, $7 = 2 \cdot 4 + (-1)$.

ii) Find the smallest positive value of $a - bq$ i.e. the largest q st. $a - bq \geq 0$.
Take q to be the largest integer less than $\frac{a}{b}$.

i) $\frac{a}{b} = \frac{2342}{55} = 42 +$ remainder. So $q = 42$.

Then $r = a - bq = 2342 - 2310 = 32$.

ii) $\frac{a}{b} = \frac{-2342}{55} = -42.\_\_ = -43 + \text{remainder}$

So $q = -43$. $r = a - bq = -2342 - 55(-43)$
$= -2342 + 2365$
$= 23$.

iii) $\frac{a}{b} = \frac{2342}{-55} = -42.\_\_$ ~~$= -43 + \text{remainder}$~~

So $q = -42$. $r = a - bq = 2342 - (-55)(-42)$
$= 2342 - 2310$
$= 32$.

3) GCD Characterization theorem 2.24:
If d is a non-negative common divisor of the integers a and b, and there exist integers x, y st. $ax + by = d$ then $\gcd(a,b) = d$.

(a) Assumptions $d \geq 0$, $d | a$, $d | b$, there exist $x, y \in \mathbb{Z}$ st. $ax + by = d$.

(b) Conclusion $\gcd(a,b) = d$.

(c) ~~By definition, the greatest co~~
If d is negative and $d | a$ and $d | b$ then also $-d | a$ and $-d | b$, and $-d > d$. So the greatest common divisor can never be a negative number.

(d) If $d=0$ then $0|a$ and $0|b$. That is, $a=q_1 0$ and $b=q_2 0$. So $a=b=0$. By definition, $\gcd(0,0)=0$. So $\gcd(a,b)=d$.

(e) Show that d divides both a and b (i.e. d is a common divisor of both ~~aa~~ and b), and for any other common divisor c of a and b, $c \leq d$.

(f) We have by assumption that $d|a$ and $d|b$. Let c be another common divisor of a and b. By 1) ii), $c \mid (ax+by)$, so ~~d~~ $c|d$. By 1) iii) $|c| \leq |d|$ i.e. $|c| \leq d$, so $c \leq d$. This finishes the proof.

4) i) To keep track in the euclidean algorithm, remember that we use the division algorithm to write ~~q~~ $a=bq+r$, always write b, q in this order, and note that we are not, for the moment, interested in q.

$a=21$, $b=15$

$$21 = 15 \times 1 + 6$$
$$15 = 6 \times 2 + 3$$
$$6 = 3 \times 2 + 0$$

So $\gcd(21, 15) = 3$.

Now substitute backwards through the previous calculation, remembering to solve for the last remainder in order to replace it by a larger number.

$$3 = 15 - 6 \times 2$$
$$= 15 - (21 - 15 \times 1) \times 2$$
$$= 15 + 15 \times 2 - 21 \times 2$$
$$3 = 15 \times 3 - 21 \times 2$$ so $x = -2$ and $y = 3$.

ii) $a = 200, \ b = -45$

$$200 = (-45)(-4) + 20$$
$$-45 = 20 \times (-3) + 15$$
$$20 = 15 \times 1 + 5$$
$$15 = 5 \times 3 + 0$$ So $\gcd(200, -45) = 5$.

$$5 = 20 - 15 \times 1$$
$$= 20 - (-45 - 20 \times \bar{3}) \times 1$$ only substitute for one number at a time!
$$= 20 \times (-2) + 45 \times 1$$
$$= [200 - (-45)(-4)] \times (-2) + 45 \times 1$$
$$= 200 \times (-2) + 45 \times 8 + 45 \times 1$$
$$5 = -200 \times 2 + 45 \times 9$$ $x = -2, \ y = 9$.

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iii) $a = 52804, \quad b = 3600$

$$52804 = 3600 \times 14 + 2404$$
$$3600 = 2404 \times 1 + 1196$$
$$2404 = 1196 \times 2 + 12$$
$$1196 = 12 \times 99 + 8$$
$$12 = 8 \times 1 + 4$$
$$8 = 4 \times 2 + 0$$

$\gcd(52804, 3600) = 4.$

$$
\begin{aligned}
4 &= 12 - 8 \times 1 \\
&= 12 - (1196 - 12 \times 99) \times 1 \\
&= 12 \times 100 - 1196 \times 1 \\
&= (2404 - 1196 \times 2)^{\times 100} - 1196 \times 1 \\
&= 2404 \times 100 - 1196 \times 201 \\
&= 2404 \times 100 - (3600 - 2404 \times 1) \times 201 \\
&= 2404 \times 301 - 3600 \times 201 \\
&= (52804 - 3600 \times 14) \times 301 - 3600 \times 201 \\
4 &= 52804 \times 301 - 3600 \times 4415
\end{aligned}
$$

$x = 301, \quad y = 4415$

5) i) The integers a and b are relatively prime if $\gcd(a,b) = 1$.

ii) $\gcd(a,b) = 1$, $\gcd(b,c) = 1$

Counterexample

Let $a = 6$, $b = 25$, $c = 9$.

Then $\gcd(a,b) = \gcd(6,25) = 1$.

$\gcd(b,c) = \gcd(25,9) = 1$.

But $\gcd(a,c) = \gcd(6,9) = 3$.

So a, and c are not relatively prime.

iii) We are given that $\gcd(a,c) = 1$ and $\gcd(b,c) = 1$.

Let d be any common divisor of ab and c.

So $d \mid ab$ and $d \mid c$.

As $\gcd(a,c) = 1$ there exist x, y st.

$$ax + cy = 1.$$

Multiply through by b: $abx + cby = b$.

In this equation, $d \mid ab$ and $d \mid c$, so d divides the left-hand side, hence $d \mid b$.

So d is a common divisor of c and b. As $\gcd(c,b) = 1$, $d = 1$. Thus $\gcd(ab, c) = 1$.

iv) Let n, $n+1$ be any two consecutive integers.
Suppose d is a common divisor of n and $n+1$.
then d divides $(n+1)-n$ i.e. d divides 1.
So $d=1$.