Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 1: prime numbers
to be completed by Monday January 12 2015

1) i) State the definition of a *prime number*.
   ii) State the definition of *relatively prime numbers*.
   iii) Prove or give a counterexample to the statement: any pair of prime numbers is relatively prime.

2) i) State the definition of the phrase $a|b$.
   ii) Let a, b be any non-zero natural numbers. Show that if $a^2 = 2b^2$ then $2|a$ and $2|b$.
   iii) Use (b) to show that you can deduce a contradiction from the assumption that $\sqrt{2}$ is a rational number.
   iv) What does this prove?
   v) Prove that $\sqrt{3}$ is irrational.

3) i) Find two consecutive prime numbers which differ by at least 5.
   ii) Find two consecutive prime numbers bigger than 100 which differ by at least 5.
   iii) Find two consecutive prime numbers which differ by at least 10.
   iv) What does it mean to say that some quanitity can be *arbitrarily large*?
   v) Prove that the gap between consecutive prime numbers can be arbitrarily large.

4) Consider the set of even positive natural numbers:

$$E = \{2, 4, 6, 8, \ldots\} = \{2n : n \in \mathbb{N}, n \neq 0\}$$

Define the relation of *divisibility relative to* E, $|_E$, by $a|_E b$ if and only if there is $q \in E$ such that $b = aq$.

   i) Find examples of numbers a, b in E such $a|_E b$ is true.
   ii) Find examples of numbers a, b in E such $a|_E b$ is false.
   iii) Define what it should mean to say p *is prime in* E.
   iv) Find a complete description of all numbers which are prime in E. Prove that your description is correct.
   v) Prove or give a counterexample to the statement: every number in E can be written as a product of numbers which are prime in E.
   vi) Prove or give a counterexample to the statement: the factorization of a number in E into factors which are prime in E is unique.