Math 1C Solutions to Practice Problems

1) Look them up!

2) (a) $\forall x \in \mathbb{R} \left(\exists y \in \mathbb{R} \, (y^3 = x) \Rightarrow x > 0 \right)$

(b) $\forall x \in \mathbb{N} \left(\exists y \in \mathbb{N} \, (x = 4y+1) \Rightarrow \exists a,b,c,d \in \mathbb{N} \, (x = a^2+b^2+c^2+d^2) \right)$

(c) $\forall n \in \mathbb{N} \left(n \geq 8 \Rightarrow \exists x,y \in \mathbb{N} \, (x,y \geq 0 \, \& \, n = 3x+5y) \right)$

(d) Fix p prime.

$$\forall r \in \mathbb{N} \left(0 < r < p \Rightarrow \exists q \in \mathbb{N} \, (r^p = qp+1) \right)$$

$$\Rightarrow \forall r \in \mathbb{N} \left(\exists q \in \mathbb{N} \, (r^p = qp+1) \right)$$

3) Prove by induction on n:

$$\forall n \geq 8 \left(\exists x,y \in \mathbb{N} \, (n = 3x+5y) \right).$$

Base case: $n = 8$ $\quad n = 3 \cdot 1 + 5 \cdot 1$

IH: $\quad n = 3x + 5y$ for some $x,y \in \mathbb{N}$.

Case 1: $y \neq 0$. Then
$$\begin{aligned} n+1 &= 3x+5y+1 \\ &= 3x+5(y-1)+5+1 \\ &= 3(x+2)+5(y-1). \end{aligned}$$

As $y \neq 0$, $y-1 \geq 0$, so $x+2$ and $y-1$ satisfy the ~~necessary~~ desired condition.

Case 2: $y=0$. Then $n=3x$, and since $n \geq 8$, $x \geq 3$. Then
$$\begin{aligned} n+1 &= 3x+1 \\ &= 3(x-3)+9+1 \\ &= 3(x-3)+5\cdot 2 \end{aligned}$$

As $x \geq 3$, $x-3 \geq 0$, so $x-3$ and 2 satisfy the desired conditions.

4) By the division algorithm, there are q and $0 \leq r_1 < p$ st. $r = qp + r_1$.

Then
$$\begin{aligned} r^p = (qp+r_1)^p &= (qp)^p + p(qp)^{p-1} r_1 + \dots \\ &\quad + p(qp) r_1^{p-1} + r_1^p \\ &= pQ + r_1^p, \text{ where } Q \in \mathbb{N} \end{aligned}$$

By assumption, since $0 \leq r_1 < p$, there is $q_1 \in \mathbb{N}$

st. $r_1^p = pq_1 + 1$.

Hence
$$r^p = pQ + pq_1 + 1$$
$$r^p = p(Q+q_1) + 1.$$

5) A, B are countably infinite; say $f : \mathbb{N} \to A$,
$g : \mathbb{N} \to B$ are bijections. Define a function
$h : \mathbb{N} \to A \times B$ as follows:

observe that any $n \in \mathbb{N}$ can be written
uniquely as $n = 2^r(2m-1)$.

$h(n) = (f(r), g(m))$. The uniqueness gives that h is well-defined.

h is onto: for any $(a,b) \in A \times B$ there exist
$r, m \in \mathbb{N}$ st. $a = f(r)$, $b = g(m)$, as f and
g are onto. Then $(a,b) = h(2^r(2m-1))$.

h is 1-1: suppose $h(n) = h(n')$. Then
$(f(r), g(m)) = (f(r'), g(m'))$, so $f(r) = f(r')$
and $g(m) = g(m')$, so $r = r'$, $m = m'$ as

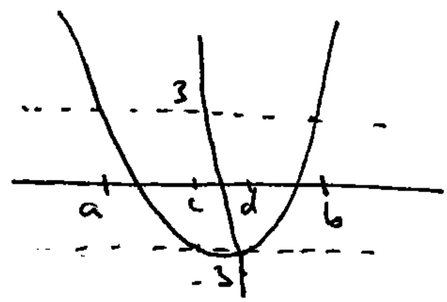
f and g are 1-1. Hence

$$n = 2^r(2m-1) = 2^{r'}(2m'-1) = n'.$$

Alternative: use Lemmas 4-6-2 and 4-6-4 and define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by

$$h(n, m) = (f(n), g(m)).$$

6) Let $f(x) = (x-1)(x+2)$



From the shape of the graph, it is clear that $|f(x)| \geq 3$ if and only if $x \leq a$ or $x \geq b$ or $c \leq x \leq d$, where a, b are the solutions to $(x-1)(x+2) = 3$

c, d are the solutions to $(x-1)(x+2) = -3$.

Use the quadratic formula in each case, to get

$b = -1 + \frac{1}{2}\sqrt{21}$, $a = -1 - \frac{1}{2}\sqrt{21}$

c, d do not exist in $\mathbb{R}$.

Hence $|(x-1)(x+2)| \geq 3$ if and only if $x \in (-\infty, -1-\frac{1}{2}\sqrt{21}) \cup (-1+\frac{1}{2}\sqrt{21}, \infty)$

5

7) Changing the problem slightly: let A be a set of size n: $A = \{a_1, a_2, \dots, a_n\}$. Prove that any function from A to any other set of size n is injective iff it is surjective.

Base case $n=1$: $f(a_1) = b_1$ is the only function, and it is a bijection.

IH: statement true for all sets of size $\leq n$.

$n+1$: $A = \{a_1, \dots, a_{n+1}\}$ Let $f : A \to B$ be any function to any set B of size $n+1$.

First suppose f is injective. Then $B' = \{f(a_1), f(a_2) \dots, f(a_n)\}$ is a subset of B with n elements. Since f is injective, $f(a_{n+1}) \notin B'$, so $B' \cup \{f(a_{n+1})\}$ has $n+1$ elements, hence equals all of B. (Notice that this argument does not use the IH.)

Now suppose f is surjective. Suppose for contradiction that $f(a_{n+1}) = f(a_n)$. Then

~~Ets~~ the range of f on $A \setminus \{a_{n+1}\}$ is the same as the range of f on A. Thus both of these sets have $n+1$ elements. But as $A \setminus \{a_{n+1}\}$ has only n elements, this is impossible. Hence f is injective.

(Notice that this argument does not use the IH either.)