Dr. M. Kolster   The Riemann Hypothesis

Formulated by Riemann in 1859.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

www.functions.wolfram.com – has a program to simulate the ζ-function.

If $s=1$, this series diverges

$s=2$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$$

$$\sum \frac{1}{n^4} = \frac{\pi^4}{190} = \zeta(4)$$

$$\sum \frac{1}{n^{2k}} = \zeta(2k) = \frac{\pi^2}{\dots} \quad \text{Euler.}$$

$\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$.

Riemann: there is a continuation of ζ which defines a function ~~on~~ all, or almost all, the complex plane.

The method of analytic continuation:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad : \text{ converges for } \mathrm{Re}(s) > 0.$$

$$\eta(s) = \zeta(s) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots\right)$$

$$= \zeta(s) - \frac{2}{2^s}\zeta(s) = \left(1 - 2^{1-s}\right)\zeta(s).$$

Thus $\zeta(s) = \dfrac{1}{1-2^{1-s}}\,\eta(s)$

We take this as the defn of $\zeta(s)$ for $Re(s) > 0$, $s \neq 1$.

Functional equation for $\zeta(s)$ – relates $\zeta(s)$ to $\zeta(1-s)$.

$Re(s) = \frac{1}{2}$ gives a line of symmetry from this functional equation.

RH: the complex zeros of $\zeta(s)$ all have real part equal to $\frac{1}{2}$.

Easy to see all should lie in $0 < Re(s) < 1$.

Ex values: $\zeta(0) = -\frac{1}{2}$ ← equiv. to $\mathbb{Z}$ has unique factorization ← 1, −1 are the numbers invertible in $\mathbb{Z}$.

$\zeta(-1) = \frac{1}{12}$

$\zeta(-2) = \zeta(-4) = \dots = 0$

trivial zeros

$$n^s = n^{x+iy} = n^x n^{iy}$$
$$= e^{x\ln(n)} e^{iy\ln(n)}$$
$$= e^{x\ln(n)}\left(\cos(y\ln(n)) + i\sin(y\ln(n))\right)$$

Connection with the Prime Number Theorem

Gauss: for $x \in \mathbb{R}, x > 0$ $\pi(x) = \#$ primes $\le x$.

$\pi(10) = 4$

$\pi(11) = 5 = \pi(12)$

$\pi(13) = 6$

π is a step function; & Gauss found a smooth function which approximates π for large x:

$$\pi(x) \sim \frac{x}{\ln(x)} \quad \text{i.e.} \quad \frac{\pi(x)}{x/\ln(x)} \longrightarrow 1 \text{ as } x \to \infty.$$

Or equivalently:
$$\frac{\left|\pi(x) - \frac{x}{\ln(x)}\right|}{x/\ln(x)} \longrightarrow 0 \text{ as } x \to \infty$$

Conjectured by Gauss, proved in 1896 by Hadamard and de Vallée Poussin.

Note: de Brange has a claim that he has proven the Riemann hypothesis.

→ Use the fact that $\zeta(s)$ has no zero on the line $\operatorname{Re}(s) = 1$.

Gauss: $\pi(x) \sim li(x) = \int_2^x \frac{1}{\ln(t)}\,dt$

– integrate by parts to see that the integral is asymptotic to $\frac{x}{\ln(x)}$.

Riemann's function $R(x)$ ~ summation of certain $\dfrac{li(x^{1/n})}{n}$

<u>Theorem</u> (Riemann) $\pi(x) = R(x) - \sum\limits_{\rho:\, \zeta(\rho)=0} R(x^{\rho})$.

This is a pretty amazing result.

Assume $\zeta(\rho)=0$. Then $\zeta(\bar{\rho})=0$.

Assume RH. $\rho = \frac{1}{2} + it$

$$x^{\rho} + x^{\bar{\rho}} = x^{\frac{1}{2}}(x^{it} + x^{-it})$$

$$= 2x^{\frac{1}{2}} \cos(t \ln(x))$$

The cosine is a wave; as t increases the frequency increases. This gives us the "music of the ~~f~~ primes".

Number the zeros: $\rho_n = \frac{1}{2} + it_n$, with $t_n > 0$.

RH satisfied for the known 10 billion zeros.

Warning: $\pi(x) < li(x)$ for $x < 10^{316}$

but ~~th~~ in fact $\pi(x)$ and $li(x)$ cross each other infinitely often.

Solution attempts.

Montgomery/Dyson: prime number distribution related to distribution of energy levels in quantum chaos, which is related to random matrix theory. (Nina Snaith, Alain Connes).