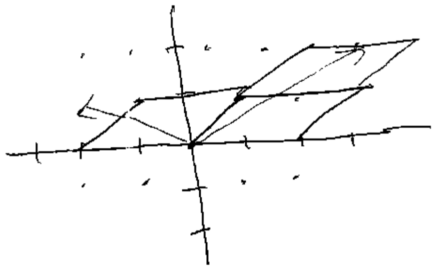
J. Hambleton **The Hodge Conjecture**

Intersection of 3 big fields: Algebra ($\mathbb{Z}$, polynomials) Analysis (functions, diffeqns)

Geometry (shapes, symmetry)



$\mathbb{R}^2 = \mathbb{C}$ integer points (m, n)

Cover the plane with a lattice;

a collection of selected integer points.

Add two chosen vectors $(3, 2), (-2, 1)$.

Q. Is every vector in the lattice an integral linear combination of the given vectors?

No.

Yes, if we allow rational coefficients.

An imaginary tale: a history of the complex numbers

$\mathbb{C}$: analysis

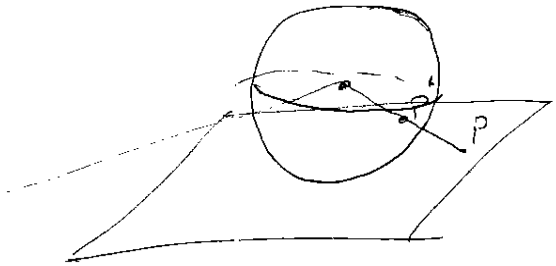
yellow lattice = geometry

red vectors: algebra

Hodge conjecture:

- analytic setting
- geometric structure embedded
- given alg. vectors — do they form a rational basis?

Big defect of $\mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$: it goes on forever!
How to correct this? Add a point at ∞ for each potential direction of parallel lines. This "line at ∞" is a circle.



equator of sphere is the line at ∞.

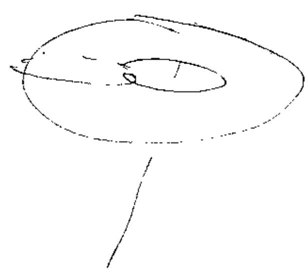
Euclidean plane maps onto lower half of sphere; antipodal points on the equator get identified with a point. This is the **real projective plane**.

Can also be identified with the set of lines in $\mathbb{R}^3$ through 0.

complex projective plane is the space of complex lines in $\mathbb{C}^3$ through the origin
$\mathbb{C}P^2$

$$\mathbb{C}P^n = \{ \text{complex lines in } \mathbb{C}^{n+1} \text{ through } 0 \}$$
$$= \{ \lambda \bar{v} : \bar{v} \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C} \setminus \{0\} \}.$$

$\mathbb{C}P^1$: identify each line with a vector of length 1. Gives $S^3 = \mathbb{R}^3 \cup \{\infty\}$



$\mathbb{C}P^1$ looks like a 2-sphere S^2.

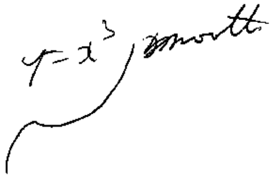
$\mathbb{CP}^N$ a closed ~~and bounded~~ universe.

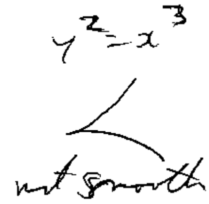
Has coordinates: $[z_0, z_1, \dots, z_N]$. Here we can do algebra and analysis.

An <u>algebraic</u> <u>variety</u> X is the zero set of d polynomials in $\mathbb{CP}^N$.

"Flatland"

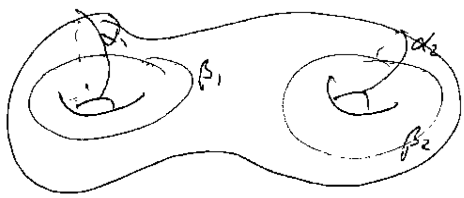
Assume the variety is smooth:





Topology: associate to X vector spaces over $\mathbb{C}$, and lattices in each $\mathbb{C}^n$.

All other curves are $\mathbb{C}$-linear combinations of $\alpha_1, \beta_1, \alpha_2, \beta_2$



Inside X, look at subvarieties $Y \subseteq X$, of $\frac{1}{2}\dim(X)$.

These act as the red vectors.

Hodge conjecture: there are enough red vectors to span the yellow vectors.