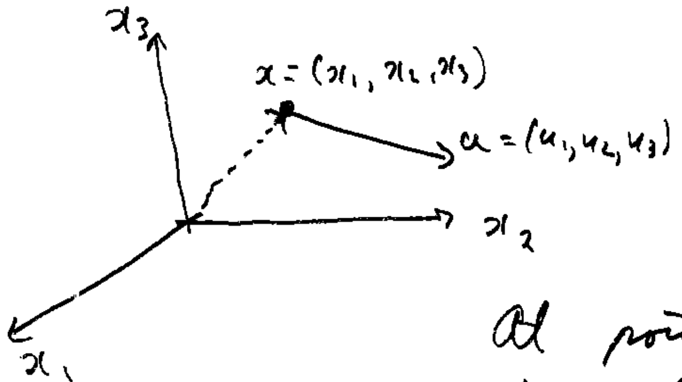
03/21/2007 16:32 IFAX scanner@mail.math.mcmaster.ca → Paula Marcoux 001/004

W. Craig

Navier - Stokes Equations

Incompressible Navier - Stokes equations



Describe a fluid by writing down a velocity vector field.

At point x, fluid is moving with velocity u.

u is a function of x; this gives us a vector field.

u is also a function of time; $u(x,t)$.

Now let $x(t)$ be the position of a fluid particle at time t.

Then $\frac{d}{dt}x(t) = \dot{x}(t) = u(x(t), t)$.

Now, if $x(t) = (x_1(t), x_2(t), x_3(t))$ then

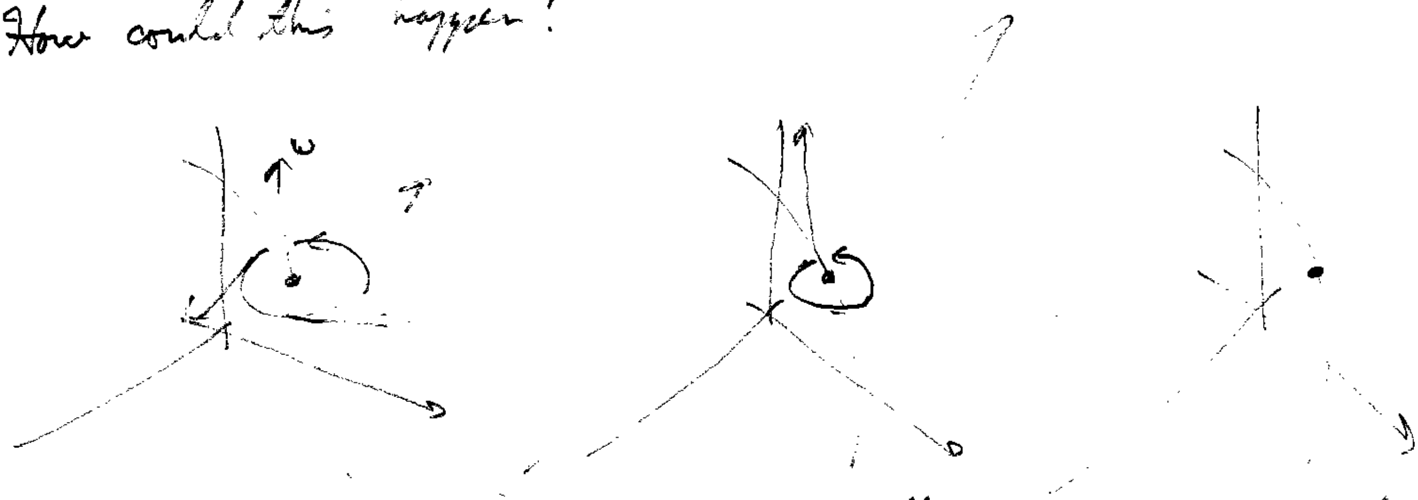
$$\dot{x}(t) = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt} \right).$$

The fluid particle moves so that the tangent to its path is given by $u(x,t)$.

2.

A singularity would be: $u(x,t) \to \infty$ as $(x,t) \to (x_0, t_0)$

How could this happen?



ω = curl = vorticity

ring gets smaller, so that velocities get larger

ring gets arbitrarily small, so vorticity goes to ∞.

Clay problem: do the eqns which are supposed to describe fluid flow ~~exhibit~~ have solutions which have singularities.

Impact outside mathematics: these equations used to model fluid flow in:

- internal combustion engines
- ocean waves / tsunamis
- astrophysics / star formation

Inside mathematics: need new models esp. for turbulence.

Mathematical choice / taste : choose problems related to the world around us.

- Navier 1822 posed the eqns
- Stokes 1840s derived them more rigorously
- Leray 1934 showed that weak solns exist
- Caffarelli, Kohn, Nirenberg 1982 bound on Hausdorff dimension of singular set.
- Craig 2006

Derive the Navier-Stokes equations

$$ma = F.$$

Set $m = 1$, by choosing the units.

$$\dot{x}(t) = u(x(t), t) \quad \text{velocity}$$

$$\ddot{x}(t) = \frac{d}{dt}\left(u(x(t), t)\right) \quad \text{acceleration}$$

$$= \partial_t u + \partial_{x_1} u \frac{dx_1}{dt} + \partial_{x_2} u \frac{dx_2}{dt} + \partial_{x_3} u \frac{dx_3}{dt}$$

$$= \partial_t u + u_1 \partial_{x_1} u + u_2 \partial_{x_2} u + u_3 \partial_{x_3} u$$

$$\ddot{x}(t) = \partial_t u + (u \cdot \partial_x) u$$

F made up of : - pressure not allowed to increase
- viscosity coefficient on the Laplacian

$$F = -\nabla p + \nu \Delta u.$$

Pressure $p(x,t)$. Constant on some level surfaces

$$\nabla p = (\partial_{x_1} p, \partial_{x_2} p, \partial_{x_3} p)$$

R. Force acts to oppose change in pressure, because the fluid is assumed incompressible — its volume cannot change: $\nabla \cdot u = 0$.