Math 1C Homework # 5

§5-2 5/ relation $\subset$ on $\mathcal{P}(\mathbb{N})$.
(in fact, the properties are the same on ~~th~~ any set of sets).

$\subset$ is not reflexive : no set is a proper subset of itself

$\subset$ is not symmetric : if $A \subset B$ and $A \neq B$ then B ~~is not $\subset$ so~~ there is an element of B which is not in A, so B is not a subset of A.

$\subset$ is transitive : suppose $A \subset B$ and $B \subset C$. then for every $a \in A$, $a \in B$. Since $a \in B$, then $a \in C$. thus $A \subset C$.

7/ set $\mathbb{N}$, relation $m R n \iff m > 10n$.

R is not reflexive: for any m, $m < 10m$.

R is not symmetric: if $m > 10n$ then $n < m$, so $n < 10m$.

R is transitive: if $m > 10n$ and $n > 10p$ then $m > n$, so $m > 10p$.

13/ set L, all straight lines in plane
relation $\perp$, perpendicular

$\perp$ not reflexive: a line is not perpendicular to itself.

$\perp$ is symmetric: clear

$\perp$ is not transitive: if $a \perp b$ then $b \perp a$,
but $\neg(a \perp a)$.

§5.3 2/ (a) relation R on $\mathbb{R} \times \mathbb{R}$ defined by

$$(x,y) \, R \, (z,w) \iff x+z \leq y+w$$

R is not reflexive: if $y < x$ then $x+x > y+y$, so $\neg\big((x,y) \, R \, (x,y)\big)$

R is symmetric: if $(x,y) \, R \, (z,w)$ then

$$x+z \leq y+w$$

so $z+x \leq w+y$

i.e. $(z,w) \, R \, (x,y)$.

R is not transitive: if $(x,y) \, R \, (z,w)$ then
$(z,w) \, R \, (x,y)$, but $\neg\big((x,y) \, R \, (x,y)\big)$.

§5.3 1a/ $A = \{1, 2, 3\}$

Consider $R = \{(1,1), (1,2)\}$.
R is not reflexive, not symmetric; is transitive.

2b/ Similar to 2a above; this relation is reflexive, is not symmetric; is transitive.

5a/ R, S are reflexive relations on A. Then $R \cup S$ is also reflexive, since for all $a \in A$, $(a,a) \in R$ and $(a,a) \in S$, so $(a,a) \in R \cup S$.

5e/ R, S are transitive relations on A. It does not follow that $R \cup S$ is transitive. For it may be that $(a,b) \in R \setminus S$, $(b,c) \in S \setminus R$ and (a,c) is in neither.

7/ By this definition of partial order, $\subset$ and $<$ are not partial orders, as they are not antisymmetric (in fact, they are asymmetric). The other properties are easy to check.

§5.5 4/ Fix $p>1$. Set $\mathbb{Z}$, relation R defined by $xRy \iff p \mid (x-y)$ or $\exists k \in \mathbb{Z}$ st $x-y=pk$.

(a)
R is reflexive: $x-x=0=p\times 0$.
R is symmetric: if $x-y=pk$ then $y-x=p(-k)$
R is transitive: if $x-y=pk$ and $y-z=pk'$ then $x-z=x-y+y-z=p(k+k')$.

(b) $[0]=\{x\in\mathbb{Z}: p\mid(x-0)\}=\{pk : k\in\mathbb{Z}\}$

$[1]=\{x\in\mathbb{Z}: p\mid(x-1)\}=\{pk+1 : k\in\mathbb{Z}\}$

$[-1]=\{x\in\mathbb{Z}: p\mid(x-(-1))\}=\{x\in\mathbb{Z}: p\mid(x+1)\}$
$=\{pk-1 ; k\in\mathbb{Z}\}$

$[p]=\{x\in\mathbb{Z}: p\mid(x-p)\}=[0]$

$[-p]=[p]=[0]$

$[p+1]=\{x\in\mathbb{Z}: p\mid(x-(p+1))\}=\{x : p\mid(x-1)\}$
$=[1]$.

(c) Show that, if $0\le r<r'<p$, then $[r]\neq[r']$.

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It suffices to show that $r' \notin [r]$, as equivalence classes are either equal or have non-empty intersection:

For any $x \in \mathbb{Z}$, $x \in [r]$ iff $p \mid (x-r)$

$$\iff x - r = pk$$

$$\iff x = pk + r.$$

Now, since $0 \leq r' \neq r$, by the division algorithm, r' can be written uniquely as $r' = p \cdot 0 + r'$. Thus $r' \neq pk + r$ for any k.

So $r' \notin [r]$.

(d) Prove that any equivalence class is one of $[r]$, ~~thus~~ $\cancel{0}$ for some $0 \leq r < p$.

By the division algorithm again, for any $x \in \mathbb{Z}$, $x = pk + r$ for some unique r with $0 \leq r < p$.

Thus $p \mid (x-r)$, so $[x] = [r]$.

8/ relation R on $\mathbb{R}\times\mathbb{R}$ defined by

$$(x,y)\ R\ (u,v) \iff x^2+y^2 = u^2+v^2.$$

(a) Since relation is based on equality, clear that R is an equivalence relation (think about it!)

(b) $[(0,0)] = \{(x,y) : x^2+y^2 = 0\} = \{(0,0)\}.$

$[(1,1)] = \{(x,y) : x^2+y^2 = 2\}$

$[(3,4)] = \{(x,y) : x^2+y^2 = 25\}$

(c) $[(a,b)] = \{(x,y) : x^2+y^2 = a^2+b^2\}$

$=$ circle center $(0,0)$, radius $\sqrt{a^2+b^2}$.

10/ $A = \{x : x \leq 0\}$, $B = \{x : x > 0\}$

Define R on $\mathbb{Z}$ by

$x R y \iff x, y$ both $\in A$ or x, y both $\in B$

$\iff (x \leq 0 \ \& \ y \leq 0) \vee (x > 0 \ \& \ y > 0)$