# Math 1C 06-07 Homework # 4

**§4.5 5** $A = \{\{1,2\}, \{1,3\}, \{2\}, \{2,5\}, \{3,4,5\}\}$

$f : A \to \mathbb{N}$ $f(a) =$ the sum of the elements of $a$.

$$f[A] = \{ f(a) : a \in A\}$$
$$= \{3, 4, 2, 7, 12\}$$

$g \circ f : A \to \mathbb{N}$ $g(n) = \begin{cases} \text{largest prime} \le n, & n \geq 1 \\ 1 & , \text{ow} \end{cases}$

$$g\circ f[A] = \{ g\circ f(a) : a \in A\}$$
$$= \{ g(3), g(4), g(2), g(7), g(12)\}$$
$$= \{3, 3, 2, 7, 11\}.$$

$g$ is not 1-1; for example $g(3) = g(4)$.

$g\circ f$ is not 1-1; $g\circ f(\{1,2\}) = g\circ f(\{1,3\})$.

9/ $f : \mathbb{R}^2 \to \mathbb{R}^2$: $f((x,y)) = (x+2y, x-y)$.

Notice that $f$ is a linear transformation, given by the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. Since $\det(A) = -1-2 \neq 0$, $A$ is invertible. By linear algebra, we know that the associated transformation is one-to-one and onto, with inverse given by $A^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}$, or $g(x,y) = \left(\frac{1}{3}(x+2y), \frac{1}{3}(x-y)\right)$

§4.6

4/ $A$ is denumerable, say $f: \mathbb{N} \to A$ is a bijection. $x \notin A$. Define $g: \mathbb{N} \to A \cup \{x\}$ by

$$g(0) = x$$
$$g(n+1) = f(n) \text{ for } n \geq 0.$$

For any $b \in A \cup \{x\}$, either $b = x$ and $b = g(0)$ or $b \neq x$, so $b = f(n)$ for some $n$, hence $b = g(n+1)$. Either way, $b \in$ range of $g$.

Let $m, n \in \mathbb{N}$, $m < n$.

If $m = 0$ then $g(m) = x \notin A$, so $g(m) \neq g(n) \in A$

If $m \neq 0$ then $g(m) = f(m-1)$
$g(n) = f(n-1)$

and $f(m-1) \neq f(n-1)$ as $f$ is 1-1.

thus $g$ is a bijection from $\mathbb{N}$ to $A \cup \{x\}$, hence $A \cup \{x\}$ is denumerable.

5/ $A, B$ both denumerable, say $f: \mathbb{N} \to A$, $g: \mathbb{N} \to B$ are bijections. $A \cap B = \emptyset$.
Define a function $h: \mathbb{N} \to A \cup B$ as follows.

$h(2n) = f(n)$
for $n \geq 0$.
$h(2n+1) = g(n)$

$h$ is onto $A \cup B$: for if $a \in A$, $a = f(n)$ for some $n \in \mathbb{N}$ as $f$ is onto, so $a = h(2n)$.
If $b \in B$ then similarly $b = g(m)$ for some $m \in \mathbb{N}$, so $b = h(2m+1)$.

$h$ is 1-1: for consider any $m \neq n$ in $\mathbb{N}$.
If $m$ is even and $n$ odd (or vice-versa) then $h(m) \in A$ and $h(n) \in B$, hence $h(m) \neq h(n)$ as $A, B$ are disjoint.
If $m, n$ both even then $h(m) = f(m/2)$, $h(n) = f(n/2)$ and $f(m/2) \neq f(n/2)$ as $f$ is 1-1.
Similarly, if $m, n$ both odd then $h(m) \neq h(n)$ because $g$ is 1-1.

8/ Assume $f: A \to \mathbb{N}$ is a 1-1 function.
Assume that $A$ is not finite. We show $A$ is denumerable by defining a function $g: A \to \mathbb{N}$

which is 1-1 and ~~not~~ onto. Write $I = f[A] \subseteq \mathbb{N}$, and Write $A = \{ a_i : i \in I \}$ (so $f(a_i) = i$).

Define $g : A \to \mathbb{N}$ by $g(a_n) =$ least $i$ such that $i \neq g(a_k)$ for any $k < n$.

$g$ is 1-1: Suppose $a_n \neq a_m$. Then $n \neq m$, as $f$ is 1-1, so we may assume $n < m$.

By definition, $g(a_m) =$ least $i$ st $i \neq g(a_k)$ for any $k < m$

In particular, $g(a_m) \neq g(a_n)$.

$g$ is onto: for suppose not. Then there is a least number $n_0 \in \mathbb{N}$ st. $n_0 \notin$ range$(g)$. Then $B = \{ a \in A : g(a) < n_0 \}$ has exactly $n_0 - 1$ elements. Since $A$ is infinite, there must be a least $k$ st. $a_k \in A \setminus B$. But then $g(a_k) = n_0$, by the definition of $g$. This contradiction shows that $g$ is onto.

Consider the function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{N}$ defined by $\varphi(r) =$ greatest integer $\leq r$. $\varphi$ is not 1-1, and $\mathbb{R}^{\geq 0}$ is not denumerable.

9/ Assume $f : \mathbb{N} \to A$ is onto, and suppose that $A$ is not finite. To show $A$ is denumerable, construct a function $g : \mathbb{N} \to A$ which is a bijection.

Consider the sets $f^{-1}[a] = \{ i \in \mathbb{N} : f(i) = a \}$.
For every $a \in A$, $f^{-1}[a]$ is non-empty, so has a least element. Define a sequence which picks out the least element of each of these sets: that is,

let $k_0 = 0$

$k_1$ = least integer $i$ s.t. $f(i) \neq f(0)$.

$k_{n+1}$ = least integer $i$ s.t. $f(i) \notin \{f(0), \dots, f(n)\}$.

The sequence $\{k_n\}$ is strictly increasing.
Now define $g(n) = f(k_n)$.

$g$ is 1-1: if $n \neq m$ then $k_n \neq k_m$; say $k_n < k_m$.
Then $f(k_m) \notin \{ f(i) : i < k_m \}$; in particular,
$f(k_m) \neq f(k_n)$. i.e. $g(n) \neq g(m)$.

$g$ is onto: for any $a \in A$, $a = f(i)$ for some $i$.
Let $k = \min f^{-1}[a]$. Then $k = k_j$ for some $j \leq k$.
then $a = f(k) = f(k_j) = g(j)$.

§4.7 2/ $A$ uncountable, $B$ countable. Prove that $A \setminus B$ is uncountable.

Suppose for contradiction that $A \setminus B$ were countable. then $A = A \setminus B \cup B$ is a union of disjoint countable sets, hence by Problem 5 of §4.6 is itself countable, which is a contradiction.

5/ $X$ is infinite set. Prove that there is no bijection between $X$ and $\mathcal{P}(X)$.

Suppose for contradiction that there is such a bijection, $f: X \to \mathcal{P}(X)$. Consider the following subset of $X$.

$x \in f(x) \to x \notin A$ & $x \notin f(x) \to x \in A$

$$A = \{ x \in X : \cancel{f(x)} \ \cancel{x \notin f(x)} \}.$$

that is, for each element $x$ of $X$, $x$ goes ~~into~~ $A$ if and only if $x \notin f(x)$.

Claim: $A$ is not in the range of $f$.

$\not$ f.a. suppose there is $y \in X$ with $f(y) = A$.

then if $y \in A$ then $y \in f(y)$, so $y \notin A$. ※

If $y \notin A$ then $y \notin f(y)$, so $y \in A$. ※

Either case leads to a contradiction, so $f$ cannot be onto.