Math 1C 06-07 Homework #3 Solutions

§3.3 1/ Let (a,b), (c,d) be two intervals (open, half-closed, closed).

Assume $(a,b) \cap (c,d) \neq \phi$, for otherwise trivial.

Case 1 $a \leq c$.

Case 1.1 $c < b \leq d$. Then $(a,b) \cap (c,d) = (c,b)$

Case 1.2 $b > d$. Then $(a,b) \cap (c,d) = (c,d)$.

Case 2 $a > c$

Case 2.1 $c < b \leq d$ Then $(a,b) \cap (c,d) = (a,b)$

Case 2.2 $b > d$. Then $(a,b) \cap (c,d) = (a,d)$.

In all cases, the intersection is an interval.

The union of two disjoint intervals is not an interval.

§3.5 1/ (a) $|5x-1| > 2 \iff 5x-1 < -2$ or $5x-1 > 2$

$5x < -1$ or $5x > 1$

$x < -\frac{1}{5}$ or $x > \frac{1}{5}$

solution set is $(-\infty, -\frac{1}{5}) \cup (\frac{1}{5}, \infty)$.

(b) $x^2 > x+2 \iff x^2 - x - 2 > 0$

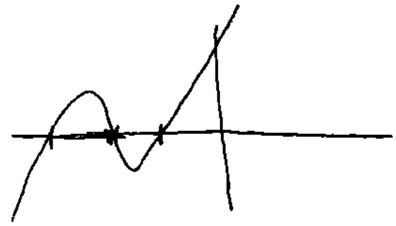
$\iff (x+1)(x-2) > 0$

$\iff x > -1$ and $x > 2$

or $x < -1$ and $x < 2$

solution set is $(-\infty, -1) \cup (2, \infty)$

(c) $(x+1)(x+2)(x+3) > 0$



solution set $(-3,-2) \cup (-1,\infty)$

(d) $\dfrac{x}{x+1} < 1$

$\cancel{\text{Case 1} \quad x+1 < 0 \quad \dfrac{x}{x+1} < 1 \iff x > x+1}$

$\cancel{\iff}$

If $x+1 < 0$ then $|x+1| < |x|$, so $\dfrac{x}{x+1} > 1$.

If $x+1 > 0$ then $|x+1| > |x|$, so $\dfrac{x}{x+1} < 1$

Thus solution set is $(-1,\infty)$.

(e) $\left|\dfrac{3x+2}{x+3}\right| > 3 \iff \dfrac{3x+2}{x+3} > 3$ or $\dfrac{3x+2}{x+3} < -3$

Solve $\dfrac{3x+2}{x+3} > 3$.

Case 1 $x+3 > 0$. $3x+2 > 3(x+3) \iff 2 > 9$ F.

no solutions

Case 2 $x+3 < 0$ $3x+2 < 3(x+3) \iff 2 < 9$ T

solution set is $\{x : x < -3\}$.

Solve $\dfrac{3x+2}{x+3} < -3$

Case 1 $x+3 > 0$ $3x+2 < -3(x+3) \iff 6x < -11$

$x < -11/6$

Case 2 $\quad x+3<0 \qquad 3x+2 > -3(x+3)$

$$\iff 6x > \cancel{\ \ } -11$$

$$x > -11/6 .$$

solution set is $(-3, -11/6)$

Hence $\left|\frac{3x+2}{x+3}\right| > 2 \iff x \in (-\infty,-3) \cup (-3,-11/6)$

§3.6 $\quad$ 3/ $\quad A_r = \{ x \in \mathbb{R} : 0 \le x < r \}$.

$$\bigcup_{r\in\mathbb{R}^+} A_r = \{ x \in \mathbb{R} : \exists r \in \mathbb{R}^+ \, (0 \le x < r) \}$$
$$= [0, \infty) .$$

$$\bigcap_{r\in\mathbb{R}} A_r = \{ x \in \mathbb{R} : \forall r \in \mathbb{R}^+ \, (0 \le x < r) \}$$
$$= \{ 0 \} .$$

4/ $\mathcal{F}$ a family of sets, let $A \in \mathcal{F}$.

Show that (1) $A \subseteq \bigcup \mathcal{F}$ for all $A \in \mathcal{F}$

(2) if X is a set with property (1) then $\bigcup \mathcal{F} \subseteq X$.

(1) Let $x \in A$. Since $\bigcup \mathcal{F} = \{ x : \exists A \in \mathcal{F} (x \in A) \}$ in particular, $x \in \bigcup \mathcal{F}$.

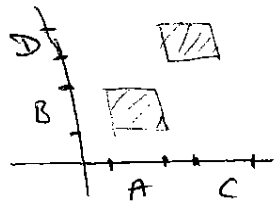
(2) Suppose $X \supseteq A$ for every $A \in \mathcal{F}$.
Let $x \in \bigcup \mathcal{F}$. Then $x \in A$ for some $A \in \mathcal{F}$, so
$x \in X$. Thus $\bigcup \mathcal{F} \subseteq X$.

5/ Claim: $\bigcap \mathcal{F}$ is the largest set X such that
$X \subseteq A$ for all $A \in \mathcal{F}$.

Pf. Let $x \in \bigcap \mathcal{F}$. Then $x \in A$ for every $A \in \mathcal{F}$,
hence $\bigcap \mathcal{F} \subseteq A$.

Suppose X is another set with the same property,
and consider $x \in X$. ~~then~~ As $X \subseteq A$ for any $A \in \mathcal{F}$,
$x \in A$. Thus $x \in \bigcap \mathcal{F}$. Hence $X \subseteq \bigcap \mathcal{F}$.

§3.7 2/ $A \times B \cup C \times D \subseteq A \cup C \times B \cup D$



Suppose $A \cap C = \phi$ and $B \cap D = \phi$.
Then $A \cup C \times B \cup D$ will have some element (a, d)
with $a \in A$ and d ~~$\in D$~~ $\notin B$. Thus $(a,d) \notin A \times B$
$a \notin C$ and $(a,d) \notin C \times D$.

Hence $A \times B \cup C \times D \neq A \cup C \times B \cup D$

6/ A has m elements, B has n elements, prove $A\times B$ has $m\times n$ elements.

Proof by induction on n. Write $B=\{b_1,\dots,b_n\}$.

<u>$n=1$</u> $A\times B = A\times\{b_1\} = \{(x,y): x\in A \;\&\; y=b_1\}$, which has m elements.

<u>IH</u> If B has n elements then $A\times B$ has $m\times n$ elements.

<u>$n+1$</u> B has $n+1$ elements, $B = B_1\cup\{b_{n+1}\}$, where B_1 has n elements.

By Theorem 3.7.1(a), $A\times B = A\times B_1 \cup A\times\{b_{n+1}\}$.

By IH, $A\times B_1$ has mn elements.

By case $n=1$, $A\times\{b_{n+1}\}$ has m elements.

The union is disjoint, hence $A\times B$ has $mn+m = m(n+1)$ elements.

<u>§4.2</u> 3/ $f:\mathbb{N}\to\mathbb{N}$

(a) f increasing $f(x)=2x$

(b) f decreasing ~~[illegible]~~ not possible, for $f(0)$ must be larger than any other value, but if $f(0)=M$ there are only finitely many values to be assigned.

(c) $m < n \to f(m) = f(n)$ f any constant function will do, say $f(x) = 1$.

(d) f increasing on odd numbers, constant on even inputs.

$$f(n) = \begin{cases} 2n, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

if m odd, n even, $m < n$ then $f(m) > f(n)$.

if m even, n odd, $m < n$ then $f(m) < f(n)$.

6/ $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x - 2$

$$g(x) = \frac{x^2 - 4}{x - 2} = \begin{cases} x - 2, & x \neq 2 \\ \text{undefined}, & x = 2 \end{cases}$$

f and g have different domains.

§4.4 6/

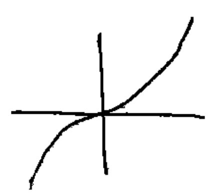
(a) $f : \mathbb{R} \to \mathbb{R}$ $f(x) = 5x + 6$.

f is 1-1, for if $f(x) = f(y)$ then $5x + 6 = 5y + 6$

$$5x = 5y$$
$$x = y.$$

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$

f is 1-1, evident from the graph



(c) $f: \mathbb{R} \to \mathbb{R}\times\mathbb{R}$ $f(x) = (x, x)$.

f is 1-1, for if $f(x) = f(y)$ then $(x,x) = (y,y)$

$\Rightarrow x = y$.

(d) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x}$

f is 1-1 as a function on $\mathbb{R}^+$.

(e) $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f(m,n) = m-n$.

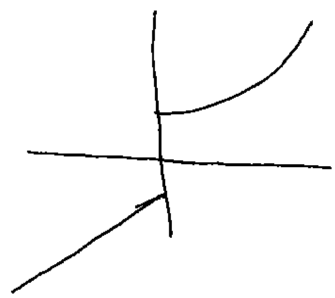
f is not 1-1 : $f(5,3) = 2 = f(4,2)$.

a, b, e are onto. a, b clear from graphs, e after a moment's thought

c is not onto

d is onto $\mathbb{R}^{\geq 0}$, but not onto $\mathbb{R}$.

7/ $f(x) = \begin{cases} x^2+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$



f is clearly 1-1 and not onto.

on $\mathbb{R}$, range of f is $(-\infty, -1) \cup [1, \infty)$

on $\mathbb{R}^+$, range of f is $(1, \infty)$.