Math 1C Homework #2

40 +8

§2.5

1g For any integer n, the number n^2+n+1 is odd.

True. $n^2+n+1 = \underbrace{n(n+1)}_{\text{even}} + 1$ odd

2

Let n be any integer. Then there exists an integer m such that either $n=2m$ or $n=2m+1$.

Case 1: $n=2m$ Then $n^2+n+1 = 4m^2+2m+1 = 2(2m^2+m)+1$ which is odd.

Case 2: $n=2m+1$ Then $n^2+n+1 = 4m^2+4m+1+2m+1+1$

$$= 2(2m^2+3m+1)+1$$

which is odd.

1h Between any two distinct rational numbers there is a third rational ~~number~~.

2

$\forall x,y$ (if x,y rational and $x<y$ then there exist z st. z is rational and $x<z<y$).

True.

Let $z = x + \frac{1}{2}(y-x)$

z is rational, as x,y both rational

$y-x>0$, so $z>x$

$\frac{1}{2}(y-x) < y-x$, so $z<y$.

1i For any real numbers x, y, if x is rational and y is
2 irrational then $x+y$ is irrational.

True.

For suppose not. Then there would be $x = \frac{p}{q}$ rational and $x+y = \frac{m}{n}$ rational.

Then $y = \frac{m}{n} - x = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq}$, which is rational.

This contradicts that y is irrational.

2 Suppose for contradiction that $\sqrt{3} = \frac{m}{n}$ is rational, and can assume m, n have no common factors.

3 Then $3 = \frac{m^2}{n^2}$ i.e. $m^2 = 3n^2$.

Claim $3 \mid m$. For if not, then $m = 3k+1$ or $3k+2$.

$$m^2 = 9k^2 + 6k + 1 \quad \text{or} \quad 9k^2 + 12k + 4$$
$$= 3(3k^2 + 2k) + 1 \quad \text{or} \quad 3(3k^2 + 4k + 1) + 1,$$

neither of which is divisible by 3.

Thus $m = 3k$. Hence
$$(3k)^2 = 3n^2$$
$$9k^2 = 3n^2$$
$$3k^2 = n^2.$$

Thus $3 \mid n^2$, hence, as above, $3 \mid n$.
This contradicts m, n having no common factors.

6 Assume: every even natural number greater than 2 is a sum of two primes.

2 Prove: every odd natural number greater than 5 is a sum of three primes.

Let m ~~2n+1~~ be an arbitrary odd number greater than 5.

$$m \cancel{2n+1} > 5 \implies m-3 > 5-3 = 2.$$

By assumption, the even number $m-3$ is a sum of two primes; ~~Then~~ $m-3 = p+q$, so $m = p+q+3$ is a sum of three primes.

7b Prove by induction: $\forall n \in \mathbb{N} \left(\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1) \right)$

3 $\underline{n=1}$ $\quad \sum_{r=1}^{n} r^2 = 1^2 = 1$

$$\frac{1}{6} 1 (1+1)(2\cdot 1+1) = \frac{1}{6} \cdot 2 \cdot 3 = 1.$$

Assume $\quad \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1).$

Prove $\quad \sum_{r=1}^{n+1} r^2 = \frac{1}{6} (n+1)(n+2)(2n+3).$

$$\sum_{r=1}^{n+1} r^2 = \sum_{r=1}^{n} r^2 + (n+1)^2$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3) + (n+1)^2 \quad \text{by IH}$$

$$= \frac{1}{6}(n+1)\left[n(2n+1) + 6(n+1)\right]$$

$$= \frac{1}{6}(n+1)(2n^2+7n+6)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3), \text{ as required.}$$

8b Prove: $\forall n \quad \sum_{i=1}^{n} (2i-1) = n^2$.

3 $n=1$: $\sum_{i=1}^{1} (2i-1) = 2\cdot 1 - 1 = 1 = 1^2$.

Assume that $\sum_{i=1}^{n} (2i-1) = n^2$.

Prove $\sum_{i=1}^{n+1} 2i-1 = (n+1)^2$.

$$\sum_{i=1}^{n+1} (2i-1) = \sum_{i=1}^{n} 2i-1 + 2(n+1)-1$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2, \text{ as required.}$$

§2.6 1(iii) Prove: if $a|b$ and $c|d$ then $ac|bd$ for $c \neq 0$.

2 Assume $b = ax$, $d = cy$ for $x, y \in \mathbb{Z}$. Then
$bd = axcy = ac(xy)$. Thus $ac|bd$.

1(vii) Prove: if $a|b$ and $a|c$ then $a|(bx+cy)$.

2 Assume $b = am$, $c = an$ for some $m, n \in \mathbb{Z}$. Then
$bx + cy = amx + any = a(mx+ny)$. Thus $a|(bx+cy)$.

6 Let $a = 2n+1$ be an odd integer.

3 Then $a(a^2-1) = (2n+1)(4n^2+4n+1-1)$
$= 4(2n+1)(n+1)n$.

For any n, either $2|(n+1)$ or $2|n$, thus $8|a(a^2-1)$.

Now, $n = 3k$ or $3k+1$ or $3k+2$.

If $n = 3k$ then $3 \mid a(a^2-1)$, as required.

If $n = 3k+1$ then $2n+1 = 2(3k+1)+1 = 6k+3$, which is divisible by 3, so $3 \mid a(a^2-1)$.

If $n = 3k+2$ then $3 \mid (n+1)$, so $3 \mid a(a^2-1)$.

Thus in each case $24 \mid a(a^2-1)$ in all cases.

§3.1 1 $\{ n \in \mathbb{N} : n > 1 \ \& \ \forall x,y \in \mathbb{N} (xy = n \Rightarrow x = 1 \vee y = 1) \}$

1 is the set of primes.

2 $P = \{ x \in \mathbb{R} : \sin(x) = 0 \}$

2 $Q = \{ \cancel{x} n\pi : n \in \mathbb{Z} \}$

$P = Q$ because $\sin(x) = 0 \iff x$ is an integer multiple of π.

9 $A = \{ x : P(x) \}$, $B = \{ x : Q(x) \}$ and

1 $\forall x \ (P(x) \Rightarrow Q(x))$.

Let $x \in A$. Then $P(x)$ holds. Thus $Q(x)$ holds, so $x \in B$. Hence $A \subseteq B$.

§3.2

(8) $A = \mathbb{R}$, $B = \mathbb{Q}$, $C = \mathbb{I}$ - irrational

(2)

$(A \cup B) \cap C = (\mathbb{R} \cup \mathbb{Q}) \cap \mathbb{I} = \mathbb{R} \cap \mathbb{I} = \mathbb{I}$

$A \cup (B \cap C) = \mathbb{R} \cup (\mathbb{Q} \cap \mathbb{I}) = \mathbb{R} \cup \phi = \mathbb{R}$.

(9) Let $A = \{ n \in \mathbb{Z} : n = 2m+1 \}$

$B = \{ n \in \mathbb{Z} : n = 2m \}$

(2) $C = \{ n \in \mathbb{Z} : n = 3m \}$

$\boxed{\begin{array}{l} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ \Rightarrow A \cap (B \cup C) \subsetneq (A \cap B) \cup C \\ \text{Indeed: let } x \in C \setminus A \\ \quad \Rightarrow x \in (A \cap B) \cup C \\ \quad\quad x \notin A \cap (B \cup C) \end{array}}$

$(A \cap B) \cup C = \{ n \in \mathbb{Z} : n = 2m+1 \ \& \ n = 2n \} \cup \{ n \in \mathbb{Z} : n = 3m \}$

$= \phi \cup C = C$.

$A \cap (B \cup C) = A \cap \{ n \in \mathbb{Z} : n = 2m \overset{\text{or}}{\cancel{\&}} n = 3k \}$

$\cancel{A \cap \{ n \in \mathbb{Z} : n = 6m}$

$= \{ n \in \mathbb{Z} : n = 2m+1 \ \& \ n = 3k \}$

$=$ odd multiples of 3.

(10) (a) Suppose $A \subseteq B$. Then if $x \in A \Rightarrow x \in B$

so $A \cup B \subseteq B$.

(8) (2) Clearly $B \subseteq A \cup B$. Thus $B = A \cup B$.

Suppose $B = A \cup B$ and let $x \in A$. Then $x \in A \cup B$,

so $x \in B$. Hence $A \subseteq B$.

(2) (b) Suppose $A \subseteq B$. Certainly $A \cap B \subseteq A$. Let $x \in A$.

Then $x \in B$ so $x \in A \cap B$. Thus $A \cap B = A$.

Suppose $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$, so $x \in B$. Thus $A \subseteq B$.

(c) Assume $A \subseteq B \cup C$ and $A \cap B = \phi$.

1 Let $x \in A$. Then $x \in B$ or $x \in C$. But $x \notin B$, hence $x \in C$. So $A \subseteq C$.

(d) Assume $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$.

1 If $x \in A$ then $x \in C$ and if $x \in B$ then $x \in C$. Thus in either case, $x \in C$, so $A \cup B \subseteq C$.

(e) Assume $A \cup B \subseteq C \cup D$, $C \subseteq A$ and $A \cap B = \phi$.

Let $x \in B$. Then $x \in A \cup B$, so $x \in C \cup D$.

2 If $x \in C$ then $x \in A$. But $A \cap B = \phi$, so this is not possible.

Thus $x \in D$. Hence $B \subseteq D$.

12 $\mathcal{P}(A \cap B)$ = set of all subsets of $A \cap B$.

Suppose $C \in \mathcal{P}(A \cap B)$. Then $C \subseteq A \cap B$, that is, if

2 $x \in C$ then $x \in A \cap B$. Thus $C \subseteq A$ and $C \subseteq B$.

So $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$ i.e. $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Suppose $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $C \subseteq A$ and $C \subseteq B$.

Thus $C \subseteq A \cap B$, so $C \in \mathcal{P}(A \cap B)$.

13 Let $C \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $C \subseteq A$ or $C \subseteq B$.

In either case, $C \in \mathcal{P}(A \cup B)$.

2 Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$.

$A \cup B = \{1, 2, 3, 4, 5\}$

$C = \{2, 5\} \in \mathcal{P}(A \cup B)$.

However $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\phi, \{1\}, \{2\}, \{1,2\}, \{5\}\}$

and $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Challenge Problems

§2.5 4 Claim: $\sqrt{n}$ is irrational iff n is not a perfect square.

2 Equivalently: $\sqrt{n}$ is rational iff n is a perfect square.

$\Leftarrow$ If n is a perfect square, then by definition $\sqrt{n}$ is an integer and hence a rational.

$\Rightarrow$ Suppose $\sqrt{n}$ is rational, and let p be a prime which divides n. Write $\sqrt{n} = \frac{m}{r}$ in lowest terms.

$$r^2 n = m^2$$

Then $p \mid m^2$, ~~If $p \nmid m$ then $m = pq + s$ for $0 < s < p$~~

$$\cancel{m^2 = p^2q^2 + 2pqs + s^2}$$
$$\cancel{= p(pq^2 + 2qs) + s^2}$$

hence $p \mid m$ (for complete detail, write m as a product of primes)

Thus $r^2 n = (pm')^2 = p^2 m'^2$.

Since m, r have no common factor, $p \nmid r$. Thus $p^2 \mid n$. Hence n is a perfect square.

§2.6 (3) Suppose n is not divisible by 3. Then

(2) $n = 3q+1$ or $n = 3q+2$.

If $n = 3q+1$ then
$n+2 = 3q+3$ is divisible by 3.

If $n = 3q+2$ then $n+4 = 3q+6$ is divisible by 3.

(4) $(3, 5, 7)$ is a prime triple.

(2) Suppose $(n, n+2, n+4)$ is any triple of numbers.
Then by 3, at least one of the three is divisible by 3.
Hence this is a prime triple only if the one which is divisible by 3 is equal to 3.

$(3, 5, 7)$ is example already given.

$(1, 3, 5)$ is not a prime triple as 1 is not prime

$(-1, 1, 3)$ is also not a prime triple.

3.1. (1)

(2)