MATHEMATICS 1AA3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University
13 March 2006

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NAME(PLEASE PRINT): SOLUTIONS

Student No.: ____________

Tutorial No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Attempt all questions. Total number of points is 50. Marks are indicated next to the problem number.

Any Casio fx991 calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED)

Write your answers in the space provided. Good luck.

Problem	Points	Mark
1	7	
2	7	
3	6	
4	12	
5	11	
6	7	
TOTAL	50	

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Table of Formulas

1) $\sin(2x) = 2\sin(x)\cos(x)$

2) $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

3) $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

4) $\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$

5) $\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$

6) $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

7) The Taylor series for the function $f(x)$ centered at a is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n.$$

If $|f^{(n+1)}(x)| \le M$ for all $|x-a| \le d$, then the nth remainder term satisfies $|R_n(x)| \le \frac{1}{(n+1)!}M|x-a|^{n+1}$.

8) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$; converges for all x.

9) $\cos(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$; converges for all x.

10) $\sin(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$; converges for all x.

11) $(1+x)^r = \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n$; converges for $|x|<1$.

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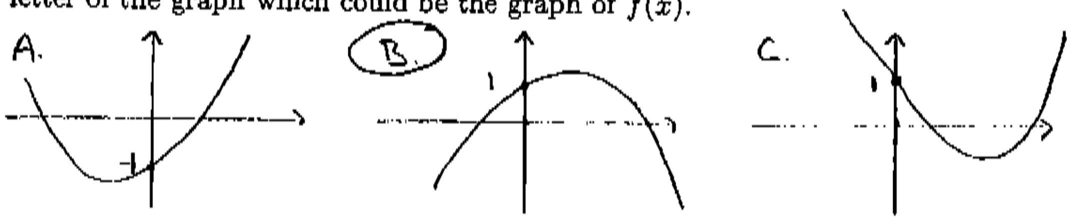
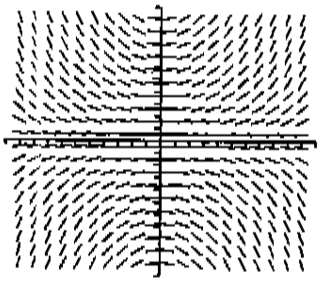
1)(a) [3] Let $T_1(x) = 1 + x$ be the first Taylor polynomial for $f(x)$ centered at 0. Circle the letter of the graph which could be the graph of $f(x)$.



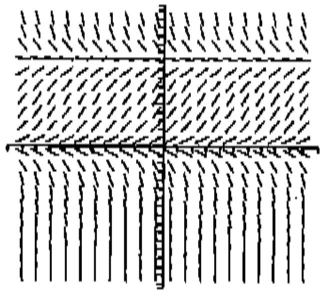
(b) [4] Next to each of the following two differential equations, indicate the letter of the correct slope field (also called direction field) from the six given.

(i) $\frac{dy}{dx} = y(10 - y)$, B

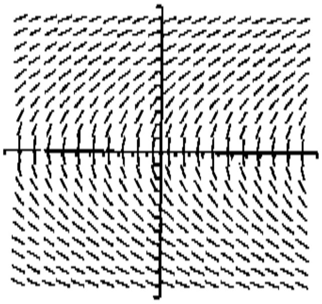
(ii) $\frac{dy}{dx} = x^2$ D



A.



B.



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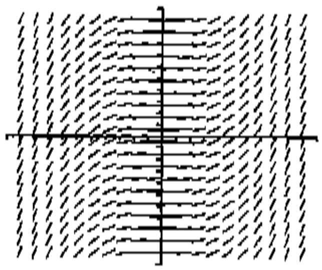

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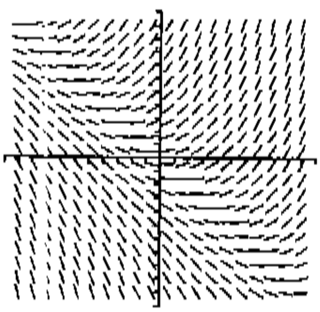

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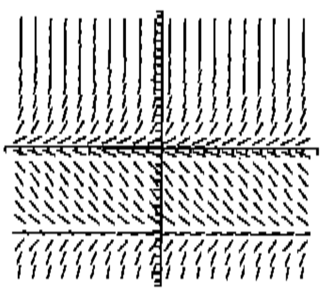

F.

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2)(a) [4] Calculate the series $\sum_{n=3}^{\infty} \frac{3^n - 1}{4^{n+1}}$.

$$\sum_{n=3}^{\infty} \frac{3^n-1}{4^{n+1}} = \sum_{n=3}^{\infty} \frac{3^n}{4^{n+1}} - \sum_{n=3}^{\infty} \frac{1}{4^{n+1}}$$

$$= \sum_{n=3}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n - \sum_{n=3}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n - \left(\frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2\right)$$

$$- \left[\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^n - \left(\frac{1}{4} + \frac{1}{4}\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right)^2\right)\right]$$

$$= \frac{1/4}{1-3/4} - \frac{1}{4}\left(\frac{29}{16}\right) - \left[\frac{1}{4}\frac{1}{1-1/4} - \frac{1}{4}\left(\frac{21}{16}\right)\right]$$

$$= \frac{5}{12}.$$

(b) [3] Show that $y = \sin(x)$ is a solution of the differential equation

$$\cos(x)y''' + \sin(x)y = -\cos(2x).$$

$y = \sin(x)$

$y' = \cos(x)$

$y'' = -\sin(x)$

$y''' = -\cos(x)$

then $\cos(x)y''' + \sin(x)y$

$= \cos(x)(-\cos(x)) + \sin(x)\sin(x)$

$= -\cos^2(x) + \sin^2(x)$

$= -\cos(2x)$ by Formulas 2,3 combined.

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3) a) [4] Find the power series expansion centered at 0 for $\int e^{x^2}\,dx$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \text{, so } e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

thus
$$\int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2n+1} x^{2n+1} + C$$

b) [2] What is the radius of convergence for the above power series? Why?

The radius of convergence for the power series for e^x is ∞; it converges for all x. Hence the power series for e^{x^2} also converges for all x. As integration preserves radius of convergence, the power series for $\int e^{x^2}dx$ ~~is~~ also converges for all x.

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4)[12] a) Find the Taylor series centered at 0 for $f(x) = \frac{1}{(3-x)^2}$.

Alternative Solution

Use binomial series to express $f(x)$ as a Taylor series or use geometric series and then differentiate. Then have to compute the derivatives for part (b).

$f(x) = (3-x)^{-2}$ $\quad f(0) = 3^{-2} = \frac{1}{9}$

$f'(x) = -2(3-x)^{-3}(-1)$ $\quad f'(0) = \frac{2}{3^3}$

$f''(x) = (-2)(-3)(3-x)^{-4}(-1)^2$ $\quad f''(0) = \frac{+2\cdot 3}{3^4}$

$f'''(x) = (-2)(-3)(-4)(3-x)^{-5}(-1)^3$ $\quad f'''(0) = \frac{+2\cdot3\cdot4}{3^5}$

$\vdots$

$f^{(n)}(x) = (-2)(-3)\ldots(-2-(n-1))(3-x)^{-2-n}(-1)^n$

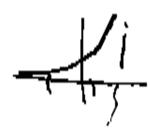
$= (-2)(-3)\ldots(-1-n)(3-x)^{-2-n}(-1)^n$ $\quad f^{(n)}(0) = \frac{(n+1)!}{3^{n+2}}$

$T(x) = \sum_{n=0}^{\infty} (n+1)\frac{1}{3^{n+2}} x^n$

b) How many terms of the Taylor series of $f(x)$ do we need to compute so that the approximation to $f(x)$ is within 10^{-2} on $[-1, 1]$?

$|R_n(x)| \le \frac{1}{(n+1)!} M\, |1|^d$, where $M \geq f^{(n+1)}(x)$ for $|x|<1$.

$f^{(n+1)}(x) = (n+2)!\,(3-x)^{-2-n-1} = \frac{(n+2)!}{(3-x)^{n+3}}$



$|f^{(n+1)}(x)| \le |f^{(n+1)}(1)| = \frac{(n+2)!}{2^{n+3}}$ for $|x|<1$

Thus $|R_n(x)| \le \frac{1}{(n+1)!}(n+2)!\frac{1}{2^{n+3}} = \frac{n+2}{2^{n+3}}$

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Find n so that $\frac{n+2}{2^{n+3}} \le 10^{-2}$.

$n=7$ $\quad \frac{n+2}{2^{n+3}} = \frac{9}{2^{10}} = 0.008 < 0.01$.

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5) a) [8] Find the power series solution to the differential equation $y'' - y = 0$.

Let $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

One solution

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

hence $c_{n+2} = \frac{c_n}{(n+2)(n+1)}$

Another solution

x^0: $2 \cdot 1 c_2 = c_0$

x^1: $3 \cdot 2 c_3 = c_1$

$\vdots$

$c_{2n} = \frac{c_0}{(2n)!}$, $c_{2n+1} = \frac{c_1}{(2n+1)!}$

Hence

$$y = c_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

b) [3] Find the solution $y(x)$ to the above differential equation which satisfies the initial conditions $y(0) = 1, y'(0) = 1$. Justify your answer.

$$1 = y(0) = c_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} 0^{2n} + c_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} 0^{2n+1} = c_0$$

$$y'(x) = c_0 \sum_{n=1}^{\infty} \frac{2n}{(2n)!} x^{2n-1} + c_1 \sum_{n=0}^{\infty} \frac{2n+1}{(2n+1)!} x^{2n}$$

$$1 = y'(0) = c_1$$

Thus
$$y = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

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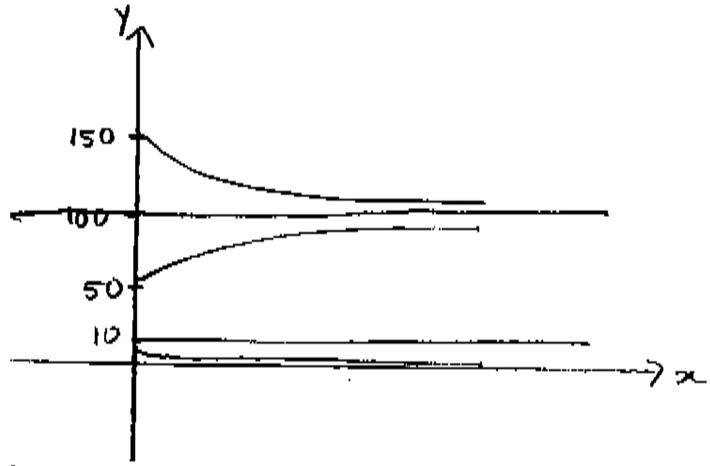
6) Consider the following differential equation:

$$\frac{dy}{dx} = (100 - y)(y - 10).$$

(a) [2] On the axes given, sketch the solutions with initial values
(i) $y(0) = 100$,
(ii) $y(0) = 10$,
(iii) $y(0) = 150$,
(iv) $y(0) = 60$,
(v) $y(0) = 5$.



(b)[5] Find the explicit solution to the differential equation with $y(0) = 20$.

$$\frac{dy}{dx} = (100-y)(y-10)$$

$$\int \frac{dy}{(100-y)(y-10)} = \int dx$$

$$\frac{1}{(100-y)(y-10)} = \frac{1/90}{100-y} + \frac{1/90}{y-10}$$

Integrating gives:

$$-\ln|100-y| + \ln|y-10| = x + C$$

$$\frac{y-10}{100-y} = e^{x+C}$$

$$y(0)=20 \Rightarrow \quad \frac{20-10}{100-20} = e^{C}$$

$$\frac{y-10}{100-y} = \frac{1}{8}e^{x}$$

$$y = \frac{100/8\, e^{x} + 10}{1 + \frac{1}{8}e^{x}}$$

THE END