McMaster University Math 1A03 Fall 2011
Midterm 1 October 13 2011
Duration: 90 minutes

Instructors: L. Barto, M. Bays, D. Haskell

Name: Solutions (DH)

Student ID Number: ______________

Instructions

– This test paper is printed on both sides of the page. There are 7 questions on pages 2 through 8; pages 9 and 10 are blank for rough work. For full credit you must show all your work.

– You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator..

– Only the McMaster standard calculator, the Casio fx 991, is permitted.

– Answers must be written in pen.

Problem	Points
1 [8]	
2 [7]	
3 [7]	
4 [7]	
5 [7]	
6 [8]	
7 [6]	
Total [50]	

1) [8 marks] No partial credit will be given on this question.

a) Find the derivative of $f(x) = e^{\tan(x)}$. Do not simplify your answer.

$f'(x) = e^{\tan(x)} \cdot \sec^2(x)$

b) Find the derivative of $f(x) = x^3 \arcsin(x)$. Do not simplify your answer.

$f'(x) = 3x^2 \arcsin(x) + x^3 \frac{1}{\sqrt{1-x^2}}$

c) If $h(x) = f(g(x))$, where $f(-3) = 11, f'(-3) = 2, f'(5) = -4, g(5) = -3$ and $g'(5) = 7$, find $h'(5)$.

$h'(x) = f'(g(x))\, g'(x)$

$h'(5) = f'(-3)\ 7$

$= 2 \times 7 = 14$

d) Let $f(x) = \int_0^x \sqrt{t^2+4}\, dt$. Find $f'(x)$.

$f'(x) = \sqrt{x^2+4}$

2) [7 marks] State and prove the Product Rule.

Product Rule If f and g are differentiable, then so is fg and $(fg)' = f'g + fg'$.

Proof $$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - fg}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}$$

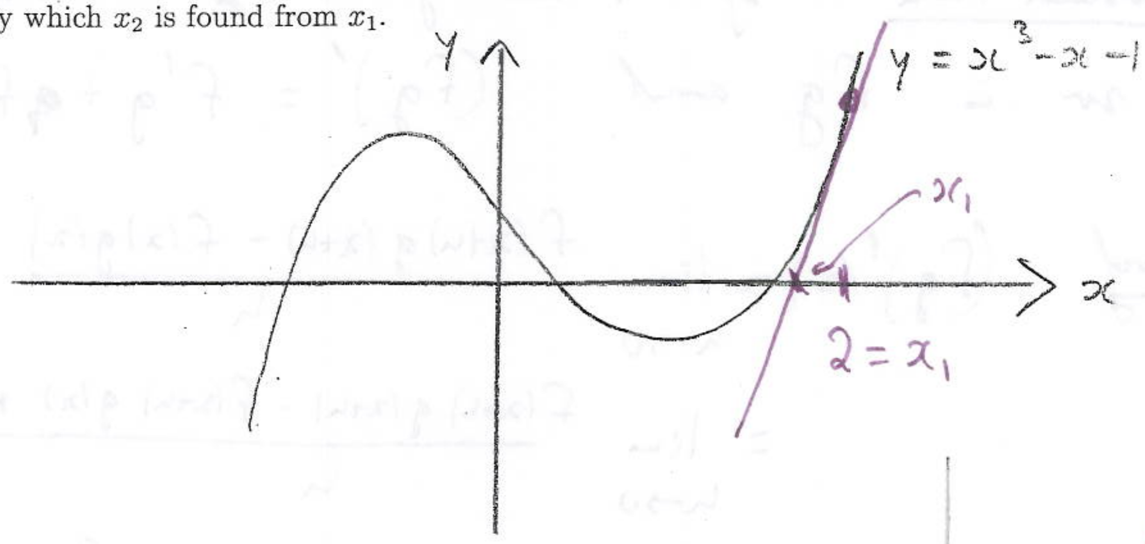
$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x)$$

as f is continuous

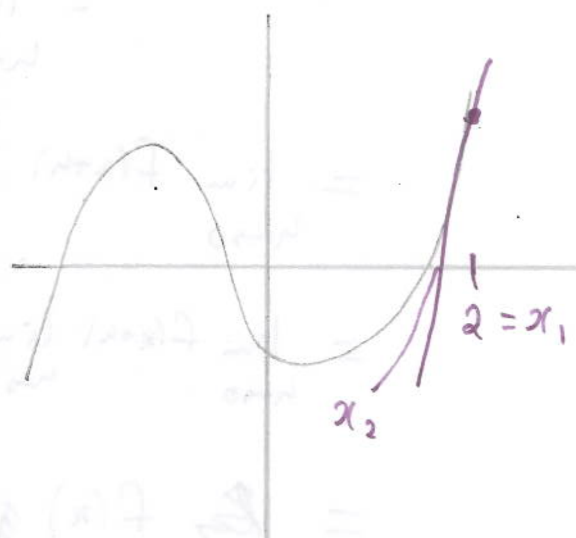
3) [7 marks] Consider the function $f(x) = x^3 - x - 1$. We aim to use Newton's method with $x_1 = 2$ to solve the equation $f(x) = 0$.

a) The sketch below shows the graph $y = f(x)$. On this sketch, indicate x_1, x_2 and the method by which x_2 is found from x_1.



b) Calculate x_2.

or



$$f(x) = x^3 - x - 1$$

$$f'(x) = 3x^2 - 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2 - \frac{8-2-1}{3(2)^2-1}$$

$$= 2 - \frac{5}{11}$$

$$\underline{x_1 = \frac{17}{11}}$$

or $f(x) = x^3 - x + 1$

$$x_2 = 2 - \frac{8-2+1}{3(2)^2-1}$$

$$= 2 - \frac{7}{11}$$

$$\underline{x_2 = \frac{15}{11}}$$

4) [7 marks]

a) State the limit definition of the derivative of the function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

b) Calculate the derivative of $f(x) = 3x^2$ using the definition you gave in a).

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

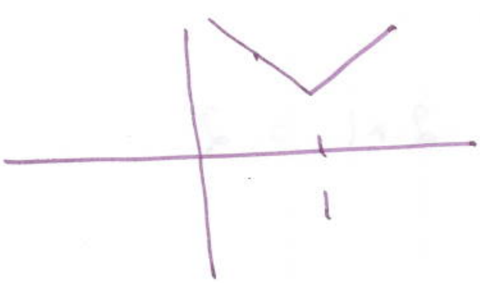
$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + h^2}{h}$$

$$= \lim_{h \to 0} (6x + h)$$

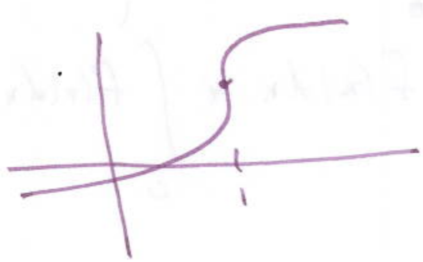
$$f'(x) = 6x$$

c) Sketch the graph of a function which is continuous but not differentiable at $a = 1$. (No need to give a formula for the function.)



or

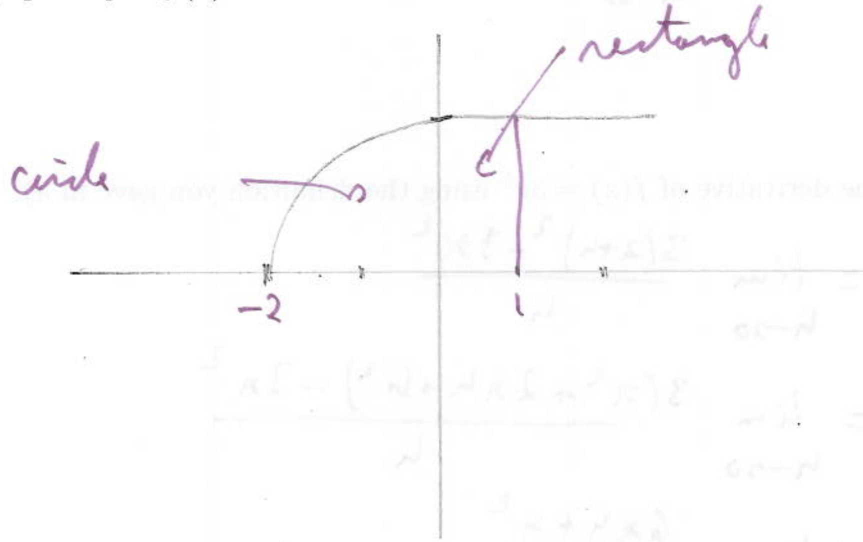
or



5) [7 marks] Consider the function

$$f(x) = \begin{cases} \sqrt{4-x^2}, & \text{if } -2 \le x \le 0, \\ 2, & \text{if } 0 \le x \le 1. \end{cases}$$

a) Sketch the graph of $y = f(x)$.



b) Find $\displaystyle\int_{-2}^{0} f(x)\,dx, \int_{0}^{1} f(x)\,dx, \int_{-2}^{1} f(x)\,dx.$

$\frac{1}{4} \times$ area of circle

$$\int_{-2}^{0} f(x)\,dx = \int_{-2}^{0} \sqrt{4-x^2}\,dx = \frac{1}{4}\pi 2^2 = \pi$$

$$\int_{0}^{1} f(x)\,dx = \int_{0}^{1} 2\,dx = 2 \times 1 = 2$$

$$\int_{-2}^{1} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = \pi + 2.$$

6) [8 marks] The following data represents the velocity of an airplane in the minutes after take-off until it reaches its cruising speed. We want to estimate the distance travelled by the airplane in these 30 minutes.

time in minutes	0	5	10	20	30
velocity in kilometers per minute	3	6.5	9	11	12.5

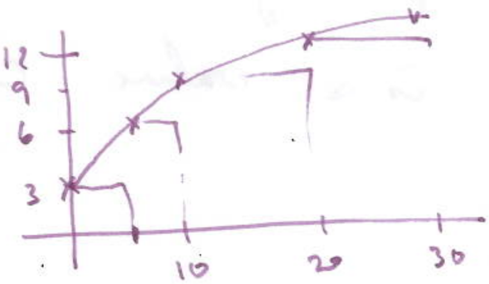
a) Estimate the distance using lefthand endpoints.

$L = 3\times 5 + 6.5\times 5 + 9\times 10 + 11\times 10$

$= 15 + 32.5 + 90 + 110$

$= 247.5$

3



b) Estimate the distance using righthand endpoints.

$R = 6\times 5 + 9\times 5 + 11\times 10 + 12.5\times 10$

$= 30 + 45 + 110 + 125$

$= 310$

3

c) Make an educated guess at the exact value of the distance, with reference to a sketch of the data to explain your reasoning.

It is clear from the picture that L is an underestimate and R is an overestimate. Thus the average of the two is a good guess: $\frac{237.5 + 310}{2} = 248.5$

2

Since the function is concave down, a slightly larger number is a better estimate - perhaps 275 km. (Assuming speed is monotone increasing between data points.)

7) [6 marks] a) State the Intermediate Value Theorem.

Suppose the function f is continuous on the interval $[a,b]$, and that $f(a) \neq f(b)$. Then for any value N between $f(a)$ and $f(b)$ there is a value $c \in (a,b)$ with $f(c) = N$.

4

2

b) Use the Intermediate Value Theorem to show that the equation $3^x = 2\cos(x)$ has a solution in the interval $(0,1)$.

Consider the function $f(x) = 3^x - 2\cos(x)$, which

4 is continuous on $(0,1)$, as exponential and the cosine are continuous.

$f(0) = \cancel{3 = 2 =} \; 3^0 - 2\cos(0) = 1 - 2 = -1$

$f(1) = 3^1 - 2\cos(1) = 3 - 2\cos(1) > 0.$

By IVT there is $c \in (0,1)$ with $f(c) = 0$,

so $3^c = 2\cos(c)$.