ArtSci 1D06 Calculus 2017–2018

Practice questions for Winter Midterm 3

This is a list of practice questions in order to prepare for Midterm 3. It represents the approximate difficulty and approximate length of the actual exam.

1)

a) State precisely what it means to say that the sequence $\{a_n\}$ diverges.

$\lim_{n \to \infty} a_n$ does not exist

b) State precisely what it means to say that the sequence $\{a_n\}$ is increasing.

for all n, $a_{n+1} \geq a_n$

c) State precisely what it means to say that the series $\sum_{n=0}^{\infty} a_n$ diverges.

$\lim_{i \to \infty} \sum_{n=0}^{i} a_n$ does not exist

d) State precisely what it means to say that the series $\sum_{n=0}^{\infty} a_n$ is bounded.

the sequence $\{a_n\}$ is bounded if there are numbers m, M such that, for all n, $m \leq a_n \leq M$.

e) State the alternating series test for convergence of the series $\sum_{n=0}^{\infty}(-1)^n b_n$, where $b_n > 0$ for all n.

If $b_{n+1} \leq b_n$ for all n

and $\lim_{n \to \infty} b_n = 0$

then the series $\sum (-1)^n b_n$ converges

2)

a) Write the formula for the Taylor series around a for a function $f(x)$.

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

b) Use your answer to a) to find the Taylor series for the function $f(x) = (1-3x)^{1/2}$ around 0. (Do not just quote a known Taylor series.)

$$f(x) = (1-3x)^{1/2} \qquad f(0) = 1$$

$$f'(x) = \left(\frac{-3}{2}\right)(1-3x)^{-1/2} \qquad f'(0) = \frac{1}{2}(-3)$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(-3)(-3)(1-3x)^{-3/2} \qquad f''(0) = \frac{1}{2}\left(-\frac{1}{2}\right)(-3)^2$$

$$f'''(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-3)(-3)(-3)(1-3x)^{-5/2} \qquad f'''(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \times (-3)^3$$

$$f^{(n)}(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)(-3)^n$$

c) Find the radius of convergence of this series.

$$T(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n-1)\right)(-3)^n x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}\frac{1}{2}\cdots\left(\frac{1}{2}-n\right)(-3)^{n+1}x^{n+1}}{\frac{1}{n!}\frac{1}{2}\cdots\left(\frac{1}{2}-(n-1)\right)(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\left(\frac{1}{2}-n\right)(-3)x\right|$$

$$= 3|x| \lim_{n\to\infty}\left|\frac{n-\frac{1}{2}}{n+1}\right| = 3|x|.$$

Series converges if $3|x|<1$ or $|x|<\frac{1}{3}$.

So radius of convergence $= \frac{1}{3}$.

3)

a) State the divergence test.

If $\lim_{n \to \infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ diverges

b) Let $\{a_n\}$ be a decreasing sequence such that $\lim_{n\to\infty} a_n = \frac{1}{2}$. Write $s_m = \sum_{n=1}^{m} a_n$. Find a lower bound for s_m (this will depend on m). Deduce that $\sum_{n=1}^{\infty} a_n$ diverges (thus verifying the divergence test for this example).

$S_1 = a_1 \geq \frac{1}{2}$ as $\{a_n\}$ is decreasing and has limit $\frac{1}{2}$.

$S_2 = a_1 + a_2 \geq \frac{1}{2} + \frac{1}{2} = 1.$

$S_m = \sum_{n=1}^{m} a_n \geq \sum_{n=1}^{m} \frac{1}{2} = \frac{1}{2} m.$

$\lim_{m \to \infty} \frac{1}{2} m = \infty$, so $\lim_{m\to\infty} S_m = \infty$,

thus $\sum_{n=1}^{\infty} a_n$ diverges by definition

4)
a) State the monotone sequence theorem.

If the sequence $\{a_n\}$ is bounded and monotonic then it converges.

b) Show that the sequence $\{ne^{-n}\}$ converges.

$a_n = ne^{-n}$. Consider the function $f(x) = xe^{-x}$ and note that $f(n) = a_n$ for all n.

$f'(x) = 1e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$.

$e^{-x} \geq 0$ for all x, $1-x \leq 0$ for $x \geq 1$, so f is decreasing for all $x > 1$.

Thus $\{a_n\}$ is decreasing.

$a_n \geq 0$ for all n.

$a_n \leq a_1$ for all n, as decreasing.

Thus the sequence is bounded above by e^{-1} and below by 0.

By the ~~b~~ monotone sequence theorem, the sequence converges.

5) The Taylor series around 0 for the function e^x is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

a) Use this to find the Taylor series around 0 for the function e^{-x^2}.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

b) Hence find a Taylor series for $\int e^{-x^2}\, dx$.

$$\int e^{-x^2}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+1} x^{2n+1} + C.$$