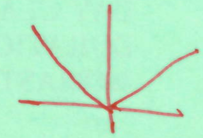
1) [10 points]

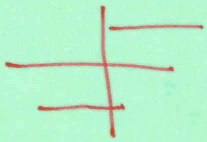
a) True or false (justify your answer): If a function is continuous on its whole domain, then it is differentiable on its whole domain.

False. Consider $f(x) = |x|$. Continuous everywhere, but not differentiable at 0



b) True or false (justify your answer): If a function is integrable on its whole domain then it is differentiable on its whole domain.

False a function like $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$



is integrable but not continuous at 0 so also not differentiable

c) Find $\dfrac{dy}{dx}$ if $x \cos y + y \cos x = 1$

$$1\cos(y) + x(-\sin(y))\frac{dy}{dx} + \frac{dy}{dx}\cos(x) + y(-\sin(x)) = 0$$

$$\frac{dy}{dx}\left(-x\sin(y) + \cos(x)\right) = y\sin(x) - \cos(y)$$

$$\frac{dy}{dx} = \frac{y\sin(x) - \cos(y)}{\cos(x) - x\sin(y)}.$$

d) Find $\int x\sqrt{x^2+1}\, dx$.

$u = x^2 + 1$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int x\sqrt{x^2+1}\, dx = \int \frac{1}{2}\sqrt{u}\, du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

2) [15 points]

a) Find $\displaystyle\int_0^\infty xe^{-2x}\,dx$.

Integrate by parts

$$u = x \qquad dv = e^{-2x}\,dx$$
$$du = dx \qquad v = -\frac{1}{2}e^{-2x}$$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$
$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2x} + C.$$

$$\int_0^\infty xe^{-2x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{t}{e^{2t}} - \frac{1}{4}\frac{1}{e^{2t}}\right) - \left(0 - \frac{1}{4}e^0\right)$$
$$= 0 + \frac{1}{4}.$$

b) Find $\displaystyle\int \frac{\sin(x)\cot(x)}{\sec(x)}\,dx$.

$$\int \frac{\sin(x)\cot(x)}{\sec(x)}\,dx = \int \sin(x)\cdot\frac{\cos(x)}{\sin(x)}\cdot\cos(x)\,dx$$
$$= \int \cos^2(x)\,dx$$
$$= \int \frac{1}{2}\left(\cos(2x)+1\right)dx$$
$$= \frac{1}{2}\left(\frac{1}{2}\sin(2x) + x\right) + C$$
$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$$

c) Find $\int \frac{6x-11}{x^2+x-12}\,dx.$

$$\frac{6x-11}{x^2+x-12} = \frac{A}{(x+4)} + \frac{B}{(x-3)}$$

$$6x-11 = A(x-3) + B(x+4)$$

$x=3$: $\quad 18-11 = B(7)$

$\underline{B=1}$

$x=-4$: $\quad -24-11 = A(-7)$

$\underline{5=A}$.

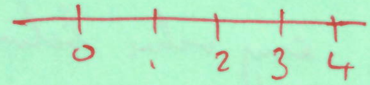
$$\int \frac{6x-11}{x^2+x-12}\,dx = \int \frac{5}{x+4} + \frac{1}{x-3}\,dx$$

$$= 5\ln|x+4| + 1\ln|x-3| + C.$$

3) [10 points]

a) Find an approximation to the integral $\int_0^4 \sqrt{7x+9}\,dx$ using a Riemann sum with left endpoints and $n = 4$.

$$L_4 = \left(f(0) + f(1) + f(2) + f(3)\right)\Delta x$$



where $f(x) = \sqrt{7x+9}$ $\qquad \Delta x = \frac{4-0}{4} = 1.$

$$L_4 = \sqrt{9} + \sqrt{16} + \sqrt{\cancel{25}\,23} + \sqrt{30}$$
$$= 3 + 4 + \sqrt{23} + \sqrt{30}$$
$$= 18.22$$

b) Find the exact value of the integral in a).

$$\int \sqrt{7x+9}\,dx = \int \sqrt{u}\,\frac{1}{7}\,du \qquad u = 7x+9, \quad du = 7\,dx$$
$$= \frac{1}{7}\cdot\frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{21}(7x+9)^{3/2} + C.$$

$$\int_0^4 \sqrt{7x+9}\,dx = \left[\frac{2}{21}(7x+9)^{3/2}\right]_0^4$$
$$= \frac{2}{21}(37)^{3/2} - \frac{2}{21}(9)^{3/2}$$
$$= \frac{2}{21}(198)$$
$$= 18.86$$

4) [5 points]

a) State the Intermediate Value Theorem.

Assume f is a continuous function on $[a,b]$. Let L be any value between $f(a)$ and $f(b)$. Then there is $c \in (a,b)$ with $f(c)=L$.

b) Is there a number that is exactly 1 less than its cube? Justify your answer.

i.e. is there a number x satisfying $x = -1 + x^3$?

That is, is there a solution to the equation

$$x^3 - x - 1 = 0\,?$$

Let $f(x) = x^3 - x - 1$. f is continuous on $(-\infty, \infty)$.

$f(0) = -1 < 0$

$f(2) = 8 - 2 - 1 > 0$.

As 0 is between $f(0)$ and $f(2)$, there is $c \in (0,2)$ with $f(c) = 0$. This c is the desired value of x.

5) [5 points] Consider the function

$$h(x) = \begin{cases} \dfrac{2\sin(x)}{x}, & \text{if } x \neq 0; \\ a, & \text{if } x = 0, \end{cases}$$

where a is a constant. Find the value of a so that h is continuous at 0.

h is continuous at 0 if $\lim_{x \to 0} h(x) = h(0)$.

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{2\sin(x)}{x}$$

$\lim_{x \to 0} 2\sin(x) = 0$

$\lim_{x \to 0} x = 0$

so L'Hospital's applies

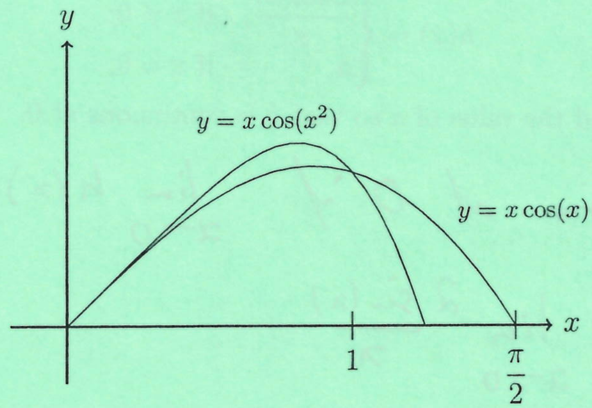
$$= \lim_{x \to 0} \frac{2\cos(x)}{1}$$

$$= 2\cos(0) = 2.$$

$h(0) = a$.

So to make h continuous, must have $a = 2$.

6) [10 points] The graphs of $y = x\cos(x^2)$ and $y = x\cos(x)$ are shown below. Find the area between them on the interval $[0, 1]$.



$$\text{area} = \int_0^1 \left(x\cos(x^2) - x\cos(x)\right) dx$$

$$\int x\cos(x^2)\,dx = \int \frac{1}{2}\cos(u)\,du \qquad u = x^2, \quad du = 2x\,dx$$

$$= \frac{1}{2}\sin(u) + C$$

$$= \frac{1}{2}\sin(x^2) + C.$$

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx \qquad u = x \quad dv = \cos(x)\,dx \qquad du = dx \quad v = \sin(x)$$

$$= x\sin(x) + \cos(x) + C$$

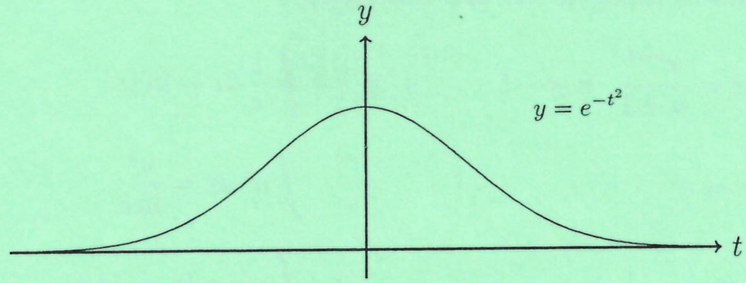
$$\text{area} = \left[\frac{1}{2}\sin(x^2) - x\sin(x) - \cos(x)\right]_0^1$$

$$= \frac{1}{2}\sin(1) - 1\sin(1) - \cos(1) - \left(\frac{1}{2}\sin(0) - 0 - \cos(0)\right)$$

$$= -\frac{1}{2}\sin(1) - \cos(1) + 1.$$

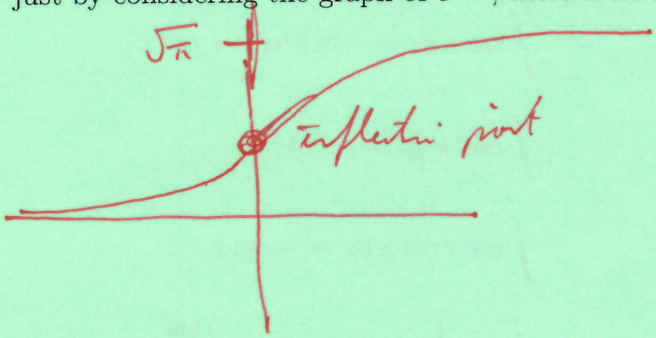
$$= 0.038926$$

7) [5 points] The function $y = e^{-t^2}$ has graph as shown below.



Consider $f(x) = \int_{-\infty}^{x} e^{-t^2}\, dt$.

a) Sketch a very rough graph of $y = f(x)$ (use the fact that $\lim_{x\to\infty} f(x) = \sqrt{\pi}$). You should do this just by considering the graph of e^{-x^2}; there is no need for any calculation.



- f is always increasing, as e^{-x^2} is never negative
- f increases most rapidly near 0, where lots of area is being added.

b) Does $f(x)$ have any critical points? If so, compute them and mark them on your graph.

$f'(x) = e^{-x^2}$ by FTC.

$f'(x) \neq 0$ for all x, so no critical points.

c) Does $f(x)$ have any inflection points? If so, compute them and mark them on your graph.

$f''(x) = -2xe^{-x^2} = 0$ when $x = 0$.

$$\left(\text{In fact, } f(0) = \int_{-\infty}^{0} e^{-t^2}\,dt \underset{\text{by symmetry}}{=} \frac{1}{2}\int_{-\infty}^{\infty} e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi} \right)$$