ArtSci 1D06 Calculus
Full year 2015–2016
Instructor: D. ~~Haskell~~

Winter Midterm – PRACTICE
Thursday 11 February ~~2016 18:45–20:15~~

Instructions There are six questions on seven pages. Answer all the questions in the space provided. If you need more paper, ask the invigilator.

NAME: Solutions

ID NUMBER:

TUTORIAL DAY AND TIME

Problem	Points
1 [10]	
2 [6]	
3 [6]	
4 [6]	
5 [6]	
6 [6]	
Total [40]	

1) [10 points]

a) State precisely what it means to say that the sequence $\{a_n\}$ diverges.

$\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n$ does not exist

b) State precisely what it means to say that the sequence $\{a_n\}$ is decreasing.

for all n, $a_{n+1} < a_n$

c) State precisely what it means to say that the series $\sum_{n=0}^{\infty} a_n$ diverges.

Let $S_m = \sum_{n=0}^{m} a_n$. the series $\sum_{n=0}^{\infty} a_n$ diverges if the sequence $\{S_m\}$ diverges

d) State precisely what is meant by the interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

the interval of convergence is the set of x such that the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges

e) State precisely what it means to say that the series $\sum_{n=0}^{\infty} a_n$ converges absolutely.

the series $\sum_{n=0}^{\infty} a_n$ converges absolutely if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

2) [6 points]

a) State the comparison test for convergence of the series $\sum_{n=0}^{\infty} a_n$.

look it up!

b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ converges.

$n^2 + 2n > n^2$, so $\frac{1}{n^2+2n} < \frac{1}{n^2}$.

$\sum \frac{1}{n^2}$ converges (p-series with $p > 1$) so $\sum \frac{1}{n^2+2n}$ also converges by the comparison test.

c) Use a partial fraction decomposition to find the exact value of the series in b).

$$\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_m = \sum_{n=1}^{m} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \dots \cancel{\frac{1}{m}} - \frac{1}{m+2}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{m+1} - \frac{1}{m+2}\right)$$

$$S_m = \frac{1}{2}\left(\frac{3}{2} - \frac{2m+3}{(m+1)(m+2)}\right)$$

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty}\left(\frac{3}{4} - \frac{1}{2}\frac{2m+3}{(m+1)(m+2)}\right) = \frac{3}{4}.$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{4}$

3) [6 points]

a) Use the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$ to find a power series representation for the function $f(x) = \frac{1}{1+x^2}$, and hence a power series representation for $g(x) = \arctan(x)$.

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ for } |x^2| < 1.$$

$$g(x) = \arctan(x) = \int \frac{1}{1+x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} + C.$$

$$\arctan(0) = 0 = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} 0^{2n+1}}_{=0} + C, \text{ so } C = 0.$$

b) What is the interval of convergence of the series for $g(x)$?

The series for $g(x)$ converges for $|x^2| < 1$ i.e. $|x| < 1$.

When $x = 1$, series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} 1^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$. This series converges by the alternating series test.

When $x = -1$, series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (-1)^{2n+1} = \sum_{n=0}^{\infty} (-1)^{3n+1} \frac{1}{2n+1} = \sum_{n=0}^{\infty} \frac{-1}{2n+1}$.

This series diverges by comparison with the harmonic series. So interval of convergence is $(-1, 1]$.

c) Deduce the exact value of the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}.$$

4) [6 points] Express the repeated decimal number

$$1.616161\ldots$$

as a fraction by summing an appropriate geometric series.

$$1.\overline{61} = 1 + 61\times 10^{-2} + 61\times 10^{-4} + 61\times 10^{-6} + \ldots$$

$$\cancel{= 1 + 61\times 10^{-2}\left(1 + (10^{-2})^{2-1} + (10^{-2})\ldots\right.}$$

$$= 1 + 61\times 10^{-2}\left(1 + 10^{-2} + 10^{-4} + 10^{-6} + \ldots\right)$$

$$= 1 + 61\times 10^{-2}\left(1 + (10^{-2})^{+1} + (10^{-2})^{2} + (10^{-2})^{3} + \ldots\right)$$

$$= 1 + \sum_{n=1}^{\infty} 61\times 10^{-2}\,(10^{-2})^{n-1}$$

This is a geometric series with $a = 61\times 10^{-2}$, $r = 10^{-2}$, so converges as $|r| < 1$.

$$= 1 + \frac{61\times 10^{-2}}{1-10^{-2}}$$

$$= 1 + \frac{61}{100}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= 1 + \frac{61}{100}\cdot\frac{100}{99}$$

$$= \frac{99+61}{99}$$

$$= \frac{160}{99}$$

5) [6 points]

a) Write the formula for the Taylor series around a for a function $f(x)$.

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

b) Use your answer to a) to find the Taylor series for the function $f(x) = (1-3x)^{1/2}$ around 0. (Do not just quote a known Taylor series.)

$$f(x) = (1-3x)^{1/2} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1-3x)^{\frac{1}{2}-1}(-3) \qquad f'(0) = \frac{1}{2}(-3)$$

$$f''(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)(1-3x)^{\frac{1}{2}-2}(-3)(-3) \qquad f''(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)(-3)^2$$

$$f'''(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(1-3x)^{\frac{1}{2}-3}(-3)(-3)(-3) \qquad f'''(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-3)^3$$

$$f^{(n)}(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n-1)\right)(1-3x)^{\frac{1}{2}-n}(-3)^n$$

$$f^{(n)}(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n-1)\right)(-3)^n$$

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n-1)\right)(-3)^n x^n$$

6) [6 points]

a) State the divergence test.

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

b) Let $\{a_n\}$ be a decreasing sequence such that $\lim_{n\to\infty} a_n = \frac{1}{2}$. Write $s_m = \sum_{n=1}^{m} a_n$. Find a lower bound for s_m (this will depend on m). Deduce that $\sum_{n=1}^{\infty} a_n$ diverges (thus verifying the divergence test for this example).

$\{a_n\}$ decreasing, so $a_{n+1} > a_n$ for all n.

$\lim_{n\to\infty} a_n = \frac{1}{2}$, so $a_n > \frac{1}{2}$ for all n.

$s_m = \sum_{n=1}^{m} a_n > \sum_{n=1}^{m} \frac{1}{2} = \cancel{2m} \frac{1}{2}m$

$\cancel{\lim_{m\to\infty} s_m = \infty}$, so $\sum_{n=1}^{\infty} a_n = \infty$ that is, the series diverges.

$\lim_{m\to\infty} s_m > \lim_{m\to\infty} \frac{1}{2}m = \infty$