McMaster University Arts and Science Winter 2013
Final Exam — PRACTICE version
Duration: 3 hours

Instructor: Dr. D. Haskell

Name: SOLUTIONS

Student ID Number: ________________

This test paper is printed on both sides of the page. There are 12 question on 12 pages. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator. More paper for rough work is available from the invigilator.

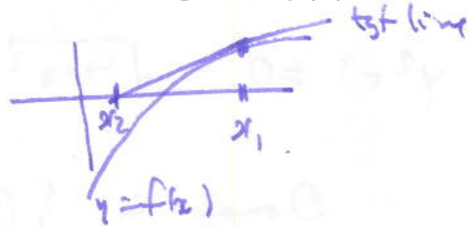
Instructions

(1) Only the standard McMaster calculator is allowed.
(2) All answers must be written in the space following the question. If you write an answer on scratch paper, indicate **clearly** where to find your answer.

This PRACTICE version of the midterm is intended to give you an idea of the format, approximate length and approximate difficulty of the actual midterm. There is no guarantee as to the actual length and difficulty of the actual exam. In particular, the actual midterm will NOT be "just the same with the numbers changed".

2) [10 points]

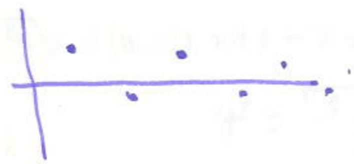
a) Sketch a graph of a function $y = f(x)$ and show how, starting from some value x_1, Newton's method is used to find an approximation x_2 to the solution of the equation $f(x) = 0$.



b) Give an example of a sequence which is bounded but not monotonic.

$\frac{(-1)^n}{n}$

bounded by 1, -1.



c) Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent or divergent. Justify your answer briefly.

p-series with $p = 2 > 1$ so converges.

d) Suppose $x(t)$, $y(t)$ are two interacting populations described by the differential equations

$$\frac{dx}{dt} = 0.01x - 0.5xy \qquad \frac{dy}{dt} = -0.01y + 0.002xy \ .$$

Which variable represents the predator population, and which is the prey? Explain briefly.

If $y = 0$, $\frac{dx}{dt} = 0.01x$ has exponential growth, so x is the prey population

If $x = 0$, $\frac{dy}{dt} = -0.01y$ will die off, so y is the predator population

e) Find $\sum_{n=1}^{100} 2$. $= 2 \sum_{n=1}^{100} 1$ ← 100 terms

$= 2\,(1 \times 100) = 200$

4) [8 points]

a) Use the fourth degree Taylor Polynomial of $\cos(2x)$ to find $\lim_{x\to 0} \frac{1-\cos(2x)}{3x^2}$.

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, converges for all x

so $\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \dots$

$$\lim_{x\to 0} \frac{1-\cos(2x)}{3x^2} = \lim_{x\to 0} \frac{1}{3x^2}\left(1 - 1 + \frac{1}{2!}(2x)^2 - \frac{1}{4!}(2x)^4\right)$$

$$= \lim_{x\to 0} \frac{1}{3}\left(\frac{1}{2!}4 - \frac{1}{4!}2^4 x^2\right) = \frac{1}{3}\cdot\frac{1}{2!}\cdot 4$$

$$= 2/3.$$

b) Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n(n+3)}(x-2)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{5^{n+1}(n+4)} \cdot \frac{5^n(n+3)}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty} |x-2| \frac{n+3}{5(n+4)} = \frac{|x-2|}{5}\lim_{n\to\infty}\frac{n+3}{n+4} = \frac{|x-2|}{5}.$$

Series converges if $|x-2| < 5$, so ~~interval of cvgce is (-3,~~ radius of cvgce is 5; ~~converges on~~

If $(x-2) = 5$ then series is $\sum \frac{(-1)^n \cancel{(x-2)} 5^n}{5^n(n+3)} = \sum \frac{(-1)^n}{n+3}$

which converges by alternating series test.

If $x-2 = -5$ then series is $\sum \frac{1}{n+3}$, which ~~converges~~ diverges by comparison with harmonic series. Thus interval of cvgce is $(-3, 7]$.

6) [8 points] Find the critical points of the function $f(x) = x^{2/3}(6-x)^{1/3}$. Classify them as local maxima or minima using the first or second derivative test. Decide if there are any absolute maxima or minima.

$$f(x) = x^{2/3}(6-x)^{1/3}$$

$$f'(x) = \frac{2}{3}x^{-1/3}(6-x)^{1/3} + x^{2/3}\frac{1}{3}(6-x)^{-2/3}(-1)$$

$$= \frac{2}{3}\cdot\frac{(6-x)^{1/3}}{x^{1/3}} - \frac{x^{2/3}}{3(6-x)^{2/3}}$$

$f'(x)=0$ when $\frac{2}{3}\frac{(6-x)^{1/3}}{x^{1/3}} = \frac{x^{2/3}}{3(6-x)^{2/3}}$

$$2(6-x)^{1/3}(6-x)^{2/3} = x^{2/3}x^{1/3}$$

$$2(6-x) = x$$

$$12-2x = x$$

$$\underline{x = 4.}$$

f' undefined when $\underline{x=0}$, $\underline{x=6}$.

$$\cancel{f''(x)=}\ f'(x) = \frac{2(6-x)^{1/3}(6-x)^{2/3} - x^{1/3}x^{2/3}}{3x^{1/3}(6-x)^{2/3}}$$

$$= \frac{12-3x}{3x^{1/3}(6-x)^{2/3}}$$

	$12-3x$	$x^{1/3}$	f'	f
$(-\infty, 0)$	+	−	−	dec
$(0, 4)$	+	+	+	inc
$(4, 6)$	−	+	−	dec
$(6, \infty)$	−	+	−	dec

$f(0)=0$ is a local min

$f(4) = 16^{1/3}2^{1/3}$ is a local max

$f(6) = 0$ is neither max nor min

None of the critical values is a ~~local~~ global max or min.

9) [8 points] Solve the differential equation: $x\ln(x)\dfrac{dy}{dx} + y = xe^x$.

$$\frac{dy}{dx} + \frac{1}{x\ln(x)}\,y = \frac{1}{\ln(x)}\,e^x.$$

F.O. linear; $\quad I(x) = e^{\int \frac{1}{x\ln(x)}\,dx}$

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln(u) + C$$
$$= \ln(\ln(x)) + C.$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$I(x) = e^{\ln(\ln(x))} = \ln(x).$$

$$\ln(x)\frac{dy}{dx} + \frac{1}{x}\,y = e^x$$

$\text{LHS} = \frac{d}{dx}\left(\ln(x)\,y\right)$, so $\quad \ln(x)\,y = \int e^x\,dx$

$$\ln(x)\,y = e^x + C$$

$$\underline{y = \ln(x)\left(e^x + C\right).}$$

11) [8 points] Decide if the following series converge absolutely.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+3}$ the absolute series is $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$

Compare with $\sum \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+3} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{n}{n+3} = 1.$$

By the limit Comparison test, either both series converge or both series diverge. $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$, so $\sum \frac{1}{\sqrt{n}}$ diverges. Hence $\sum \frac{\sqrt{n}}{n+3}$ diverges, so the original series does not converge absolutely.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ the absolute series is $\frac{|\sin(n)|}{n^3}$.

As $|\sin(n)| \le 1$ for all n,

$$\frac{|\sin(n)|}{n^3} \le \frac{1}{n^3}.$$

$\sum \frac{1}{n^3}$ converges (p-series with $p = 3 > 1$), so by the comparison test, $\sum \frac{|\sin(n)|}{n^3}$ converges also.