Arts & Science 1D06 Quiz #7 29 January 2013, 1:30pm

Full Name: SOLUTIONS Student # : ______________

TA: ______________________________

1. Evaluate the following integrals.

[3 marks] (a) $\int x\sin(x)\,dx$

Let $u = x \quad dv = \sin x\,dx$
$du = dx \quad v = -\cos x$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C$

[4 marks] (b) $\int x\arctan(x^2)\,dx$

Let $u = \arctan(x^2)$, $dv = x\,dx$
$du = \dfrac{2x}{1+x^4}$ $v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2\arctan(x^2) - \int \frac{1}{2}x^2\frac{2x}{1+x^4}\,dx$

$= \frac{1}{2}x^2\arctan(x^2) - \frac{1}{\cancel{2}}\cdot\cancel{2}\int\frac{x^3}{1+x^4}\,dx$

let $w = 1+x^4$
$dw = 4x^3\,dx$
$\frac{1}{4}\,dw = x^3\,dx$

$= \frac{1}{2}x^2\arctan(x^2) - \frac{1}{4}\int\frac{1}{w}\,dw$

$= \frac{1}{2}x^2\arctan(x^2) - \frac{1}{4}\ln|w| + C$

$= \frac{1}{2}x^2\arctan(x^2) - \frac{1}{4}\ln(1+x^4) + C$

[4 marks] (c) $\int_0^2 \frac{x}{3-x}\,dx$

Let $u = 3-x \Rightarrow x = 3-u$ when $x=0,\ u=3$

$du = -dx$ $\qquad x=2,\ u=1$

$$= \int_{u=3}^{u=1} \frac{3-u}{u}(-du)$$

$$= -\int_{u=3}^{u=1}\left(\frac{3}{u}-1\right)du = \int_1^3\left(\frac{3}{u}-1\right)du = 3\ln|u| - u\Big|_1^3$$

$$= 3\ln 3 - 3 - (3\ln 1 - 1)$$

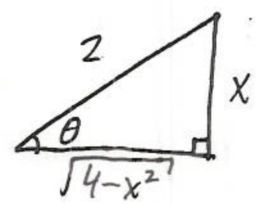
$$= 3\ln 3 - 2$$

[4 marks] (d) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$



$$= \int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\cdot 2\cos\theta\,d\theta$$

$$= \int \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\cdot 2\cos\theta\,d\theta$$

$\frac{x}{2} = \sin\theta$ so $\theta = \arcsin\left(\frac{x}{2}\right)$

$$= \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)$

$= \frac{x}{2}\sqrt{4-x^2}$

$$= \int 4\sin^2\theta\,d\theta$$

$$= \int 4\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 2\int(1-\cos 2\theta)\,d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= 2\theta - \sin 2\theta + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - \frac{x}{2}\left(\sqrt{4-x^2}\right) + C$$

[1 mark] BONUS If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that $\int_0^a f(x)g''(x)\,dx - \int_0^a f''(x)g(x)\,dx = f(a)g'(a) - f'(a)g(a)$.

① $\displaystyle\int_0^a f(x)g''(x)\,dx = f(x)g'(x)\Big|_0^a - \int_0^a f'(x)g'(x)\,dx$

② $\displaystyle\int_0^a f''(x)g(x)\,dx = f'(x)g(x)\Big|_0^a - \int_0^a f'(x)g'(x)\,dx$

① - ② :

$$\int_0^a f(x)g''(x)\,dx - \int_0^a f''(x)g(x)\,dx = f(x)g'(x)\Big|_0^a - \int_0^a f'(x)g'(x)\,dx - \left(f'(x)g(x)\Big|_0^a - \int_0^a f'(x)g'(x)\,dx\right)$$

$$= f(x)g'(x)\Big|_0^a - f'(x)g(x)\Big|_0^a$$

$$= f(a)g'(a) - \cancelto{0}{f(0)}g'(0) - f'(a)g(a) + f'(0)\cancelto{0}{g(0)}$$

$$= f(a)g'(a) - f'(a)g(a)$$

QED.

2. (a) Decompose the following fraction.

$$\frac{7x-8}{x^2-3x-4}$$

[3 marks]

$$\frac{7x-8}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$

so $7x-8 = (A+B)x + (A-4B)$

$7 = A+B \quad , \quad A-4B = -8$

$\Rightarrow A = 7-B$ sub in $7-B-4B = -8$

$-5B = -15$

$\boxed{B=3} \Rightarrow \boxed{A=4}$ thus $\frac{7x-8}{x^2-3x-4} = \frac{4}{x-4} + \frac{3}{x+1}$

(b) Find the following integral.

$$\int \frac{7x-8}{x^2-3x-4}\,dx$$

[2 marks]

$$\int \frac{7x-8}{x^2-3x-4}\,dx = \int \left(\frac{4}{x-4} + \frac{3}{x+1}\right)dx$$

$$= \int \frac{4}{x-4}\,dx + \int \frac{3}{x+1}\,dx$$

$$= 4\ln|x-4| + 3\ln|x+1| + C$$