Arts & Science 1D06 Quiz #6 15 January 2013, 1:30pm

Full Name: Solutions Student # :

TA: DH

[3 marks] 1. Evaluate $\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-x^2}}\, dx$. Give an exact answer.

$$\int_0^{1/2} \frac{2}{\sqrt{1-x^2}}\, dx = 2\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= 2\left[\arcsin(x)\right]_0^{1/2}$$

$$= 2\arcsin\left(\tfrac{1}{2}\right) - 2\arcsin(0)$$

$$= 2\frac{\pi}{6} - 2\cdot 0 = \frac{\pi}{3}.$$

[4 marks] 2. Let $g(x) = \int_{-x}^{x} \sqrt{1+t^2}\, dt$. Find $g'(-1)$.

$$g(x) = \int_{-x}^{0} \sqrt{1+t^2}\, dt + \int_0^x \sqrt{1+t^2}\, dt$$

$$= -\int_0^{-x} \sqrt{1+t^2}\, dt + \int_0^x \sqrt{1+t^2}\, dt$$

$$g'(x) = -\sqrt{1+(-x)^2}\,(-1) + \sqrt{1+x^2}$$

by FTC and chain rule.

$$g'(-1) = +\sqrt{1+(1)^2} + \sqrt{1+1^2}$$

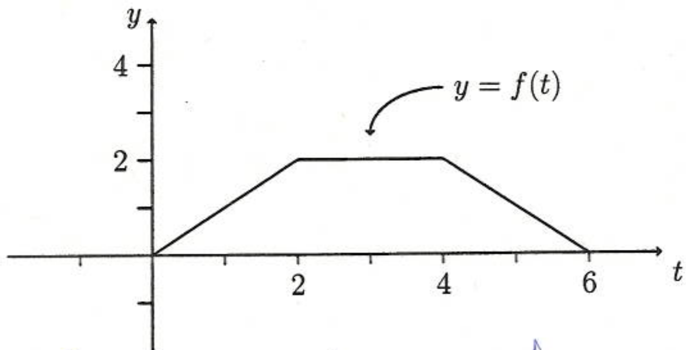
$$= 2\sqrt{2}.$$

or observe that $\sqrt{1+t^2}$ is an even function, so

$$\int_{-x}^{x} \sqrt{1+t^2}\, dt = 2\int_0^x \sqrt{1+t^2}\, dt$$

and apply FTC to find the derivative.

[3 marks] 4. Calculate $g(4)$, for $g(x) = \int_0^x f(t)\,dt$, where the graph of $f(t)$, $0 \le t \le 6$, is sketched below. What is $g'(4)$ equal to?



$$g(4) = \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt$$

$$= \text{area } \triangle + \text{area } \square$$

$$= \tfrac{1}{2}\cdot 2\cdot 2 + 2\cdot 2$$

$$= 6.$$

$$g'(4) = f(4) = 2.$$

[1 mark] BONUS Let $h(x) = \int_0^{x^2} \cos(x+t)\,dt$. Find an expression for $h'(x)$.

Observe that x is constant as t varies, So

$$\int \cos(x+t)\,dt = \sin(x+t) + C \quad (\text{substitute } u = x+t, \text{ if you like})$$

$$\text{so} \int_0^{x^2} \cos(x+t)\,dt = \Big[\sin(x+t)\Big]_{t=0}^{t=x^2} = \sin(x+x^2) - \sin(x).$$

Now differentiate w.r.t. x.

Or write $\cos(x+t) = \cos(x)\cos(t) - \sin(x)\sin(t)$, and factor the constants $\cos(x)$ and $\sin(x)$ out of the integral.

3. Use the Substitution Rule to evaluate the following integrals.

[3 marks] (a) $\int \frac{x^2}{(1+x^3)^4}\,dx$.

let $u = 1+x^3$

$du = 3x^2\,dx$

$$\int \frac{x^2}{(1+x^3)^4}\,dx = \int \frac{\frac{1}{3}\,du}{u^4} = \frac{1}{3}\left(\frac{-1}{3}\right)u^{-3} + C$$

$$= -\frac{1}{9}\,\frac{1}{(1+x^3)^4} + C$$

[4 marks] (b) $\int_1^{e^3} \frac{\ln\sqrt{x}}{x}\,dx$.

let $u = \ln(\sqrt{x})$

$\frac{du}{dx} = \frac{1}{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}$

$du = \frac{1}{2}\cdot\frac{1}{x}\,dx$

when $x=1$; $u=\ln(1)=0$

$x=e^3$, $u=\ln(e^{3/2})=\frac{3}{2}$

$$\int_1^{e^3} \frac{\ln(\sqrt{x})}{x}\,dx = \int_{u=0}^{3/2} u\cdot 2\,du$$

$$= \left[u^2\right]_0^{3/2}$$

$$= \frac{9}{4}.$$

or observe that $\ln(\sqrt{x}) = \frac{1}{2}\ln(x)$ and substitute $u = \ln(x)$

[3 marks] (c) $\int \sec^2(x)\tan(x)\,dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$\int \sec^2(x)\tan(x)\,dx$$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2(x) + C$$

or $u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$\int \sec(x)\sec(x)\tan(x)\,dx = \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\sec^2(x) + C$$

observe that these are the same answer using a trig identity