[Final Project of Math 799] VALUING AMERICAN OPTIONS WITH JUMP DIFFUSION PROCESS

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INTRODUCTION

American Option allows the holder to exercise during the life of it. The holder can choose to exercise when the option is in the money before the expire date. This feature makes the American option being path-dependent. There are some methods to value this kind of options. The widely used one is by simulating the underlying path and evaluate the option by some algorithms. Here we will introduce two methods to value American options. One is Least-Square approach by Francis A. Longstaff and Eduardo S. Schwartz. Another method is binomial method by J.C. Cox et al. First we will give basic introductions of the algorithms of both methods, then we will run simulations based on these two methods and compare the results of them.

1. A SIMPLE LEAST-SQUARE APPROACH

The Least-Square method is developed by Francis A. Longstaff and Eduardo S. Schwartz in 2001. The intuition behind this approach is using least-square method to simulate the future conditional expect return without exercise and compare the payoff with early exercise. So given the simulated underlying process, they developed an algorithm to decide exercise or not. Then based on the exercising decision the discounted final payoff will be our fair price of the option.

This approach is easy to implement since nothing more than simple lease squares is required. First we give a very simple example to show how does it work. We will use the example from the paper by Longstaff and Schwartz.

Consider an American put option on a share on non-dividend-paying stock. The put option is exercisable at a strike price of 1.10 at times 1, 2, and 3, where time three is the final expiration date of the option. The risk neutral rate r is 6%. We will illustrate the algorithm using only eight sample paths for the price of the stock. These sample paths

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Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34
TAB.	1. St	tock P	rice P	aths

are generated under the risk-neutral measure and are shown in the following matrix.

Our objective is to solve for the stopping rule that maximizes the value of the option at each point along each path. We need to calculate the cash flow on each intermediate step and decide whether to exercise the option immediately or continue the option's life until the final expiration date at time 3. The cash flow at time 3 is given below.

Path	t=1	t=2	t=3
1	—	—	0.00
2	—	_	0.00
3	—	_	0.07
4	—	_	0.18
5	—	_	0.00
6	—	_	0.20
7	—	_	0.09
8	_	_	0.00
<u>a</u> a	1 0	1	• • •

TAB. 2. Cash-flow matrix at time 3

From the stock price matrix, there are only five paths for which the option is in the money at time 2. Let X denote the stock prices at time 2 for these five paths and Y denote the corresponding discounted cash flows received at time 3 if the put is not exercised at time 2. It is equal to payoff $\times exp(-rt) =$ payoff $\times 0.94176$, where t is the time step. We use only in-the-money paths since it allows us to better estimate the significantly improves the efficiency of the algorithm. The vectors X and Y are given by the nondashed entries below.

To estimate the expected cash flow from continuing the option's life conditional on the stock price at time 2, we regress Y on a constant,

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Path	Υ	Х
1	$.00 \times .94176$	1.08
2	_	—
3	$.07 \times .94176$	1.07
4	$.18 \times .94176$	0.97
5	—	—
6	$.20 \times .94176$	0.77
7	$.09 \times .94176$	0.84
8	_	—
TAB. 3.	Regression at	time 2

X, and X^2 . The resulting conditional expectation function is $E[Y \mid X] = -1.070 + 2.983X - 1.813X^2$.

With this conditional expectation function, we now compare the value of immediate exercise at time 2, given in the first column below, with the value from continuation, given in the column below.

Path	Exercise	Continuation
1	0.02	0.369
2	—	—
3	0.03	0.0461
4	0.13	0.1176
5	—	—
6	0.33	0.1520
7	0.26	0.1565
8	—	—

TAB. 4. Optimal early exercise decision at time 2

By comparing the value in these two columns we can see that it is optimal to exercise the option at time 2 for the fourth, sixth, and seventh paths. Then we will adjust our cash flow matrix at time 2 as follows.

Proceeding recursively, we next examine whether the option should be exercised at time 1. After we did the same calculation as previous, we get that the conditional expectation function at time 1 is $E[Y \mid X] = 2.038 - 3.335X + 1.356X^2$. Then we can get the optimal early exercise decision for time 1. Finally we combine the decision for 2 time steps we get the following option cash flow matrix.

Then what we need to do is to discount each cash flow in the option cash flow matrix back to time zero and averaging over all paths.

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Path	t=1	t=2	t=3
1	—	0.00	0.00
2	—	0.00	0.00
3	—	0.00	0.07
4	—	0.13	0.00
5	_	0.00	0.00
6	_	0.33	0.00
7	_	0.26	0.00
8	—	0.00	0.00

TAB. 5. Cash-flow matrix at time 2

Path	t=1	t=2	t=3
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00
 0 0	1.	1 0	

TAB. 6. Option cash flow matrix

By this simple example, we know how to evaluate American type option now. So let us price an American put option on a share of stock, where the risk-neutral stock price process follows the stochastic differential equation

(1.1)
$$dS = rSdt + \sigma SdZ$$

and where r and σ are constants, Z is a standard Brownian motion, and the stock does not pay dividends. Then we can run Least-Square algorithm on simulated path matrix to get the price of American put options. The following table shows the simulated price for a put option with difference initial stock price, volatilities, and maturity time. Given one can exercise 50 times during the option's life, strike price 40, and risk-free rate 6%. We ran 10000 times simulation to get the option price and the early exercise value is the difference between American put price and European put Black-Sholes price.

Now let us apply this algorithm to the jump-diffusion model. The basic jump-to-ruin model presented by Merton (1976) is defined as

S	σ	T	Early Exercise Value	American Put
36	.20	1	.3744	4.2187
36	.20	2	.6005	4.3635
36	.40	1	.5889	7.3003
36	.40	2	.8062	8.5062
38	.20	1	.3992	3.2511
38	.20	2	.5983	3.5889
38	.40	1	.5964	6.4307
38	.40	2	.7932	7.7720
40	.20	1	.3364	2.4028
40	.20	2	.5029	2.8588
40	.40	1	.5523	5.6119
40	.40	2	.7674	7.0934
42	.20	1	.2454	1.7099
42	.20	2	.3963	2.2377
42	.40	1	.4763	4.8550
42	.40	2	.6922	6.4278
44	.20	1	.1708	1.1877
44	.20	2	.2992	1.7284
44	.40	1	.3956	4.1784
44	.40	2	.6071	5.8091

TAB. 7. American Put Option

follows:

(1.2)
$$dS = (r + \lambda)Sdt + \sigma SdZ - Sdq$$

where q is an independent Poisson process with intensity λ . When a Poisson event occurs, the value of q jumps from zero to one, implying dq = 1, and the stock price jumps from S to zero. Tt is not so reasonable in our case since we are pricing a put option. If the stock price goes to zero the price of the put option will increase dramatically. So we want to use a more reasonable model as following:

(1.3)
$$dS = (r+\lambda)Sdt + \sigma SdZ - \xi Sdq$$

where ξ is the jump size. So when the jump happens, the stock price jumps from S to $(1 - \xi)S$. By solving this SDE, we get: (1.4)

$$S_{t+\Delta t} = \begin{cases} (1-\xi)S_t \exp\left((r+\lambda-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_t\right) & dq = 1, \\ S_t \exp\left((r+\lambda-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_t\right) & dq = 0. \end{cases}$$

Then we can generate the sample path by simulation and apply the

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FIG. 1. Generated Paths with $\xi = \pm 1$

Least-Square algorithm to get the option price. The figure 1 and 2 are the generated paths for the jump-diffusion process with different parameters.



FIG. 2. Generated Paths with $\xi = \pm 0.5$



FIG. 3. American Option Put Price with Jump by L-S Method

Figure 4 shows the change of option prices while we change the jump size. As $\xi < 0$, it means positive jump. So the option price is very low when there exists positive jumps. As $\xi > 0$, it means negative jump. The option price increases as the jump size increases.

2. BINOMIAL METHOD

In this section we will introduce the well known method in pricing options: Binomial Method. It is first introduced by J.C Cox et al in 70's. It's main idea is to divide the time period into countable steps and use binomial tree to represent the stock process. So by this means, the stock price S will either go up to uS with u > 1 for the next time step or dS with d < 1. To allow this model work more efficiently, we need to make our binomial tree a recombine tree, which means Sud = Sdu = S. After we constructed the binomial tree, we need to set some parameters correctly in order to eliminate arbitrage chance. Since the stock price can only go up or down we need to choose the probabilities p_u and p_d respectively. Here we get the risk-neutral probability:

$$p = \frac{e^{-r\delta t} - d}{u - d}$$

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Then how can we select suitable values for u and d? On the binomial lattice we have

(2.1)
$$Var(S_1) = E[S_1^2] - E^2[S_1]$$
$$= pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$
$$= e^{r\delta t}(u+d) - 1 - e^{2r\delta t},$$

where we have used the expression for the risk-neutral probabilities and we have used the condition ud = 1. So we should solve the nonlinear equation

$$e^{r\delta t}(u+d) - 1 - e^{2r\delta t} = \sigma^2 \delta t$$

By neglecting terms in $(\delta t)^2$ and higher, an approximate solution is

$$u = e^{\sigma\sqrt{\delta t}}, d = 1/u.$$

Once we got u, d, p we can construct the binomial tree and the European option price will be given by the following formulas:

(2.2)
$$f_{i,j} = e^{-r\delta t} [pf_{i+1,j+1} + (1-p)f_{i+1,j}]$$
$$P = f(1,1).$$

where f(i, j) is the payoff at node (i, j). So from the equation above, the option price will be the value at first node. This means binomial method is working backward to find the value of the option.

For American options, we just need to compare the early exercise payoff and the continuation payoff at each node. Then the discounted back value will be the fair price. The following table shows the corresponding prices calculated by Binomial method.

By comparing table 7 and table 8, we see that the simulated result by these two different methods are quite similar. However these results are based on the general stock processes. As we see from the previous section, Least-Square algorithm can also be applied to jump-diffusion model. So how about Binomial method? In the following part we will discuss this problem.

Suppose, in the same set up of the previous tree, we instead set

$$u = e^{\sigma\sqrt{t/N}}, d = e^{\xi(t/N)}, p = \lambda(t/N).$$

This correspondence captures the essence of a pure jump process. As before, ξ is the jump size and λ is the jump rate. Then by adjusting the parameters of the no jump binomial tree, we can calculate the option price with jump process. Since ξ is used in the downward jump of the stock price, it has to be given negative values. It can be shown that the stock price movements will converge to a log-Poisson process instead of a lognormal distribution as $n \to \infty$.

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S	σ	T	Early Exercise Value	American Put
36	.20	1	.6425	4.4868
36	.20	2	1.0850	4.8480
36	.40	1	.3980	7.1094
36	.40	2	.8136	8.5136
38	.20	1	.4052	3.2571
38	.20	2	.7606	3.7512
38	.40	1	.3198	6.1541
38	.40	2	.6975	7.6763
40	.20	1	.2592	2.3193
40	.20	2	.5336	2.8895
40	.40	1	.2579	5.3175
40	.40	2	.5965	6.9225
42	.20	1	.1568	1.6213
42	.20	2	.3757	2.2170
42	.40	1	.2093	4.5880
42	.40	2	.5159	6.2515
44	.20	1	.0965	1.1134
44	.20	2	.2644	1.6936
44	.40	1	.1708	3.9536
44	.40	2	.4461	5.6481

TAB. 8. American Put Option-Binomial Method

Figure 4 shows values of American put options with different ξ values. From the graph we can see when ξ approaches zero, which means the jump size reduces, the price of put option reduces. This satisfies the property of put options.

3. CONCLUSION

In this report, we introduced two basic methods to evaluate American type options. We are interested in put options only since we know for no dividend stocks the price of American call options are the same with European calls. We compared results from these two methods for log-normal stock processes. They produced very similar results which means these two methods are working well for pricing American put options. And we calculated option prices for stock prices that follow jump processes by both methods. However, we were not able to compare the results for these settings for the following reasons: 1. The jump processes we were using for these two methods are different; 2. Although we used the same parameters in both methods, we could not XUPING ZHANG



FIG. 4. American Option Put Price with Jump by Binomial Method

compare them directly since they were used in different ways. But by the graphs we have shown we can see that the option price satisfies the property of the put option: when the downward jump increases the option worths more, when the upward jump increases the option worths less. Therefore we can conclude that these two methods are all working in different ways to solve our problem.