Business V703 Financial Modeling Valuation

Lecture 9: Game theory and probability

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Adam Smith: individual x society

- In Wealth of Nations, Adam Smith writes that: "Every individual necessarily labours to render the annual revenue of the society as great as he can. He generally neither intends to promote the public interest, nor knows how much he is promoting it... he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for society that it was no part of his intention. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.
- His larger goal was to achieve societal good, and his main concern was to show that this does not necessarily require charitable action.
- His unfortunate generalization was to conclude (as above) that self–interest necessarily leads to societal good.

John Nash: individual x society

- As we have seen, a set of individual strategies is in a Nash equilibrium if no player can unilaterally improve his outcome, given the strategies adopted by the other players.
- In this way, there are many examples (such as the prisoner's dilemma) in which each player will do what is best for himself, in direct conflict for what is best for the group.
- In other words, Nash showed that Adam Smith's conclusion was false, since sometimes self–interest leads to societal losses.
- Now compare this with the following scene from the movie A Beautiful Mind:
  http://www.haverford.edu/math/lbutler/GoverningDynamics.html

Pure x Mixed Strategies

- When the actions of a player depend on probabilities, we say that the player is following a mixed strategy. Otherwise the strategy is called pure.
- For example, the solutions to the PD presented in the previous lecture involved only pure strategies.
- When considering only pure strategies, it is possible to find games with no NE (and therefore no solution at all).
- By contrast, games with mixed strategies always have a solution.
- Mixed strategies can be introduced due to external uncertainties, trembling hands, imperfect information or a combination of all of these factors.

The Fugitive-Hunter Problem

- As an example (Ross 2006), consider the problem mentioned earlier of a fugitive who has to cross one of three bridges: (1) a safe one, (2) a rocky one, or (3) one inhabited by snakes.
- Assume further that a hunter waits for the fugitive on the other side of the river and needs to decide by which bridge to wait.
- Suppose that crossing the safe bridge is indeed perfectly safe, whereas there is a 10% probability of being hit by a rock on the rocky bridge and an 20% probability of dying of a snakebite on the cobra bridge.
- Finally, assume that the fugitive only cares about surviving, while the hunter only cares about reporting him dead. In other words, neither player cares about the way in which the fugitive might die.

FH problem in Strategic-form

- Let us say that, if the fugitive escapes, his payoff is 1 and the hunter's payoff is 0, whereas if the fugitive dies (for whatever reason), his payoff is 0 and the hunter's payoff is 1.
- In this way, we can analyze this game in strategic form as follows:

		Hunter: Safe	Hunter: Rocks	Hunter: Snakes
Fugitive	Safe	(0,1)	(1,0)	(1,0)
Fugitive	Rocks	(0.9,0.1)	(0,1)	(0.9,0.1)
Fugitive	Snakes	(0.8,0.2)	(0.8,0.2)	(0,1)

- We then observe that there are neither dominant/dominated rows or columns nor Nash equilibria for this problem.

Mixed Strategies and Indifference

- From our previous discussion of the FH problem (i.e, if for some reason the fugitive thinks one bridge is optimal, the same reason would lead the hunter to the same conclusion, therefore rendering this bridge the worst possible choice for the fugitive...) we expect that the only way to obtain a stable solution for this problem is by letting the players choose their strategies at random.
- Moreover, each player might try to choose probabilities in order to minimize the expected payoff for the opponent.
- A general principle for this is for each player to choose probabilities in such a way that the other player becomes indifferent between the choices.

The indifference solution to the FH problem

- As an example of this indifference principle, let $s_1 = 1$, $s_2 = 0.9$ and $s_3 = 0.8$ be the probabilities of a safe crossing in each of the three bridges (as determined by historical observation).
- Now let w_1, w_2, w_3 be the probabilities that the hunter will be waiting in each of the three bridges, so that $w_1 + w_2 + w_3 = 1$.
- Then the fugitive will be indifferent between the bridges provided that

$$s_1(1 - w_1) = s_2(1 - w_2) = s_3(1 - w_3).$$

- These equations can now be simultaneously solved, yielding

$$w_1 = \frac{49}{121}, \quad w_2 = \frac{41}{121}, \quad w_3 = \frac{31}{121}.$$

- Notice how the hunter assigns a higher probability to the safest bridge.

FH problem (cont.)

- Similarly, let f_1, f_2, f_3 denote the probabilities that the fugitive will choose each of the three bridges, so that $f_1 + f_2 + f_3 = 1$.
- The fugitive will reason that the hunter will be indifferent between the bridges provided that

$$s_1 f_1 = s_2 f_2 = s_3 f_3.$$

- The solution to this equation is then

$$f_1 = \frac{36}{121}, \quad f_2 = \frac{40}{121}, \quad f_3 = \frac{45}{121}.$$

- Note how the fugitive chooses the riskier bridge with higher probability !
- Such strategies correspond to a NE, since any other choice of probabilities by one player will give an advantage to the other (by introducing a difference in the expected payoffs for the bridges).

Imperfect information on trees

- The extensive-form analysis presented in the previous lecture (trees) and the corresponding Zermelo algorithm (backward induction), as well as the notion of SPE (subgame perfect equilibrium) were well suited for sequential games of perfect information.
- When simultaneous moves are allowed and/or a player needs to make a decision without full information regarding the previous moves in the game, the tree representation, Zermelo's algorithm and SPE need to be modified accordingly.
- To begin with, we need to introduce the notion of information sets, represented by ovals around nodes where the player has the same information, despite being reached by distinct paths.
- Next, Zermelo's algorithm needs to be replaced by a conditional probability method.
- Finally, the notion of SPE is replaced by that of SE (sequential equilibrium).

Selten's Horse

- We illustrate all these concepts with the example known as Selten's Horse (Kreps 1990) [see tree in the blackboard!]
- We can verify that (D, U, D) is a NE, since if player A choses 'down', then player B cannot improve his payoff by changing from 'up' to 'down', because he will not be allowed to play.
- But suppose that we start the game at node 2. Then given that player C will choose 'down', then player B can improve his outcome by choosing 'down' as well, in which case player A should choose 'up'.
- Now (U, D, D) is clearly not a NE (in either the original or the subgame), since player C can then switch to 'up', therefore sending player B back to 'up'. We are then led to the new solution (U, U, U), which is also a NE.
- We want a criterion that will select (U, U, U) as more sensible than (D, U, D).

Conditional Probabilities and Sequential Equilibrium

- First observe that we cannot use Zermelo's algorithm to select the preferred NE in Selten's horse, since player C needs to make a decision without knowing if she is at node 3 or 4.
- An alternative way to say this is to observe that the only subgame for Selten's horse is the entire game itself, so the concept of subgame perfect equilibrium (SPE) does not lead to a preferred solution.
- Instead, we replace Zermelo's algorithm by a procedure involving the conditional probabilities of being on each node.
- Similarly, the concept of SPE is replaced by that of sequential equilibrium.

Strategies and beliefs

- Let β denote the collective set of beliefs that each player has regarding the conditional probabilities for each node of each information set.
- Let Σ denote the collective set of strategies adopted by each player given their personal beliefs.
- Define a sequential equilibrium as a pair (Σ, β) with the property that β is consistent with Bayes's rule and such that, starting from each information set, each player plays optimally from then on (based on both the available strategies and beliefs of other players).

Back to Selten's Horse

- Returning to Selten's horse, suppose that player C believes to be at node 4 with probability 0.5. Then her expected payoff for playing D is $(0.5 \times 4 + 0.5 \times 0) = 2$, whereas her expected payoff for playing U is $(0.5 \times 1 + 0.5 \times 2) = 1.5$. She will therefore prefer to play 'down', in which case B must play 'down'.
- Therefore, although (D, U, D) is a NE, it is not a SE, based on a subgame starting at node 2. But neither is (U, U, U), since a subgame starting at the information set 3-4 would lead C to choose 'down'. Since these are the only two possible NE in this game, we conclude that there is no SE solution.
- Alternatively, changing the probabilities to 0.1 for node 3 and 0.9 for node 4 lead to an expect payoff for C moving 'down' equal to 0.4, versus an expected payoff for her moving 'up' equal to 1.9. In this case, (D, U, D) is clearly not a SE, whereas (U, U, U) satisfies the conditions of being a SE.

Another example of SE

- Consider now the following example from Kreps (1990) [refer to tree in the blackboard !]
- Assigning probabilities 0.7 and 0.3 to nodes 1 and 2, respectively, we see that (L, L, L) is not a SE (since 2.6 is less than 3.1)).
- If we now say that player C chooses L with probability 0.5, then (L, L, L) becomes a SE (since 2.6 is more than 2.2)
- Finally, keeping player C as it was before (i.e with no probabilities), but changing the probabilities of nodes 1 and 2 to 0.9 and 0.1, respectively, would also turn (L, L, L) into a SE (since 2 is more than 1.7).