Business V703 Financial Modeling

Lecture 5: Valuing the investment on a project

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A simple model for the project value

- Assume that the total value of a project follows a geometric Brownian motion

$$dV_t = \alpha V_t dt + \sigma V_t dz_t, \tag{1}$$

  where α and σ are positive constants and z_t is a standard Wiener process for a probability P.
- The current value $V_0 > 0$ of the project is know. Future values V_t are lognormally distributed with mean and variance increasing linearly in time.
- Ignores possibility of temporary shut down or complete abandonment.
- Project value remains positive.
- We consider a finite time interval $[0, T]$.

Good old tree approximation

- Recall now our binomial tree approximation with parameters

$$p = \frac{1 + \alpha\Delta t - d}{u - d}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}. \tag{2}$$

- This can be rewritten as

$$V_{j+1} = V_j e^{\sigma\kappa_j\sqrt{\Delta t}} \approx V_j(1 + \sigma\kappa_j\sqrt{\Delta t}) \tag{3}$$

  where

$$\kappa_j = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } 1 - p \end{cases}$$

- We then obtain an approximation based only on $\sqrt{\Delta t}$ increments but with the correct properties, since (up to order Δt) we have that

$$\begin{aligned} E[\sigma\kappa_j\sqrt{\Delta t}] &= \alpha\Delta t \\ \mathrm{Var}[\sigma\kappa_j\sqrt{\Delta t}] &= \sigma^2\Delta t. \end{aligned}$$

From a tree to a rectangular grid

- We see from the previous discussion that each trajectory of a project values following the GBM model (1) can be approximated by one particular path on a multi–period binomial tree with parameters (2).
- Instead of considering just one tree originating from a single point V_0, we can envisage several superposed trees starting from neighbouring points of the form

$$V_0 e^{\pm k\sigma\sqrt{\Delta t}}, \quad k = 1, 2, \ldots$$

- If we then chop these superposed trees at sufficiently high and sufficiently low starting points, what we obtain is a rectangular grid of project values with m rows (with $i = 1, \ldots m$) and $(n+1)$ columns (with $j = 1, \ldots, n+1$).

Project Values on the Rectangular Grid

- Effectively, the grid will consist of $(n+1)$ repeated columns, each containing the following project values:

$$V(i) = V(\bar{m})e^{(\bar{m}-i)\sigma\sqrt{\Delta t}}, \qquad i = 1, \ldots m,$$

  where $\bar{m} = \frac{m+1}{2}$ corresponds to the middle row.
- That is, the top row consists of the highest possible project value, while as we move down each column the project values decrease by multiplicative increments of size $e^{-\sigma\sqrt{\Delta t}}$.
- We might as well add a row in the bottom of the grid with the lower bound for project values, that is, $V(m+1) = 0$.
- Our final grid then has dimension $(m+1) \times (n+1)$.

A model for the investment opportunity

- Let $I > 0$ be the sunk cost of investing in the project described by (1).
- Investment in the project is an opportunity, not an obligation.
- Assume that the decision to invest is irreversible and can be taken at any time $\tau \in [0, T]$ in the future.
- The pay-off for investing at time τ is $(V_\tau - I)^+$.
- Therefore the option to invest is equivalent to a call option on the project value with strike price I.

The spanning asset assumption

- We want to value the option to invest using the techniques of option pricing.
- Need to use replication and arbitrage arguments.
- In financial mathematics this only make sense if the underlying asset is trade in the market.
- Therefore we assume that the project value is perfectly correlated to a traded financial asset.
- That is, we assume that

$$dX_t = \mu X_t dt + \alpha_X X_t dz_t, \tag{4}$$

describe the price of the so called spanning asset.

The cost of waiting

- According to the Capital Asset Pricing Model (CAPM), the equilibrium rates of return for the traded asset X_t and the project V_t should satisfy

$$\frac{\mu - r}{\sigma_X} = \frac{\bar{\alpha} - r}{\sigma} = \rho\lambda, \tag{5}$$

  where λ is the market price of risk for the economy and β is the correlation between X_t (and thereofre V_t) and the market portfolio.
- We define the difference between the equilibrium rate $\bar{\alpha}$ and the actual rate α as the below equilibrium shortfall rate $\delta = \bar{\alpha} - \alpha$, which we assume to be positive.
- When $\sigma_X = \sigma$, this reduces to $\delta = \mu - \alpha$, which corresponds to a "convenience yield" generated by having invested in the project, such as a cash–flow rate.
- Therefore δ represents the cost of delaying the project.
- This should be compared to a call option on a stock that pay no dividends.

Valuing the option to invest

- We can now value the option to invest as a call option with strike price I on an underlying asset following a geometric Brownian with drift μ and volatility σ and paying a constant dividend yield δ.
- If we denote by $F(i,j)$ the value of the option to invest at time t_j when the underlying project value is $V(i)$, then we can calculate it everywhere on the grid using the formula

$$F(i,j) = \max\{e^{-r\Delta t}[qF(i+1,j+1)+(1-q)F(i-1,j+1)], V(i)-I].$$

- In the formula above, we need to use the risk–neutral measure

$$q = \frac{1+(r-\delta)\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

Boundary values

- For the formula above to work, we need to specify the values of the option to invest at the boundaries of the grid.
- At the bottom of the grid it is clear that the option to invest should be worthless, since the project itself is not worth anything.
- Therefore, we set $F(m+1, j) = 0$ for all $j = 1, \ldots, n+1$.
- Conversely, at the top of the grid the project value should be much larger than both the sunk cost and the value of waiting, so that investment should certainly be the preferred policy.
- This leads to $F(1, j) = V(1) - I$ for all $j = 1, \ldots, n+1$.
- Finally, at the final time, there is no more opportunity to wait, so investment should be made whenever the project value is greater than the sunk cost I.
- Therefore $F(i, n+1) = (V(i) - I)^+$, for $i = 1, \ldots, m$.

Investment threshold

- The values $F(i,j)$ obtained on the grid correspond to the value of the option to invest when time equals t_j and the project value equals $V(i)$.
- That is $F(i,j) = F(V(i), t_j)$, which is the discrete version of the function $F(V,t)$.
- If we fix a particular time $t = t_j$, then we can obtain a graph of the option to invest as a function of the underlying project value at that time, that is $F_t(V)$.
- This graph has the typical shape of a convex function lying above the piecewise linear function $(V - I)^+$.
- As the project value increase, $F_t(V)$ approaches the line $(V - I)$.
- The two graphs then "smoothly paste" at a point V_t^*.
- This point (which depends on time only) is called the investment threshold.

The investment rule

- After we determine the investment threshold V_t^* as a function of t, our investment rules reduces to comparing it with the observed project value V_t at any time t.
- Explicitly, if at time t we have that $V_t < V_t^*$, then it is because $V_t - I < F(t, V_t)$ and we should not invest.
- Conversely, if at time t we have that $V_t \geq V_t^*$, then it is because $V_t - I = F(t, V_t)$ and we should invest.
- This should be contrasted with the Marshallian rule, according to which investment should occur whenever the NPV, given in this case by $V_t - I$ is positive.
- In other words, the exercise threshold in the simple NPV rule is just the sunk cost I.

Stationarity

- As you can observe by running several examples in a spreadsheet, the function $F(i,j)$ becomes stationary (that is, independent of j) as we move away from the maturity date T.
- As a result, the exercise threshold V_t^* tends to a constant (independent of time) V^*.
- This is the motivation behind the popular method of valuing an option to invest as a perpetual call option.
- Such method (not pursued here) leads one to deduce that the (necessarily time–independent) investment threshold is given by the analytical formula

$$V^* = \frac{\beta}{\beta - 1} I, \tag{6}$$

where

$$\beta = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left[\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2R}{\sigma^2}}.$$

Properties of the investment rule

- First observe that $V^* > I$, so our investment decision will be always delayed in comparison to an investor following a NPV rule.
- By running our discrete–time valuation for several different parameters (or by analyzing the expression (6) if we want to be slick), we can observe the following properties of the exercise threshold:
- $\frac{\partial V^*}{\partial I} > 0$ (lower sunk, more investment).
- $\frac{\partial V^*}{\partial \sigma} > 0$ (higher volatility, less investment).
- $\frac{\partial V^*}{\partial \delta} < 0$ (higher cash-flows, more investment).
- $\frac{\partial V^*}{\partial r} > 0$ (higher interest rates, less investment).