Business V703 Financial Modeling

Lecture 4 - Dividends and option pricing

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American calls on non-dividend-paying stock: a one-period binomial model argument

- Suppose that u and d are such that $uS_0 > K$ and $dS_0 < K$ (since the other cases are trivial).
- Let c_0 be the continuation value for the call option at time zero, obtained by risk-neutral expectations, and let $e_0 = (S_0 - K)^+$ be its exercise value at time zero.
- It is then easy to see that

$$
\begin{aligned}
c_0 &= \frac{1}{1+R} q(uS_0 - K) \\
&= \frac{1}{1+R}[quS_0 + (1-q)dS_0] - \frac{1}{1+R}[qK + (1-q)dS_0] \\
&> \frac{1}{1+R}[quS_0 + (1-q)dS_0] - \frac{1}{1+R}K \\
&= S_0 - \frac{1}{1+R}K > S_0 - K \geq e_0.
\end{aligned}
$$

- Therefore it is never optimal to exercise such option prior to maturity, confirming what we had already concluded based on

Incorporating dividends to the one-period model

- We now consider the case where the stock pays a known dividend yield D at time $t = 0$.
- That is, if the cum-dividend price of the stock at time $t = 0$ is S_0, then the owner of one share receives a dividend equal to DS_0.
- After the dividend is paid, the stock drops to its ex-dividend price $(1 - D)S_0$
- Finally, at time $T = 1$, the stock either rises to uS_0 with probability p or drops to dS_0 with probability $(1 - p)$.
- As before, the specific value of p will be irrelevant for option prices, whereas the values for u and d will need to be determined in order to match the continuous time distribution of S_0, which we will do later.

Replicating portfolio for a call option with dividends

- As before, let us try to replicate the option by holding a shares and invest b dollars in the bank at time zero.
- Accounting for the presence of dividends, we have that the portfolio replicates the call option at time $T = 1$ if and only if

$$\begin{aligned} a(uS_0 + (1+R)DS_0) + b(1+R) &= c^u := (uS_0 - K)^+ \\ a(dS_0 + (1+R)DS_0) + b(1+R) &= c^d := (dS^0 - K)^+ \end{aligned}$$

- Solving this equations for a and b gives

$$a = \frac{c^u - c^d}{(u-d)S_0}$$

and

$$b = \frac{1}{1+R}\frac{uc^d - dc^u}{(u-d)} - D\frac{c^u - c^d}{(u-d)}.$$

Pricing a European call with dividends

- Having found a self-financing replicating portfolio, the law of one price dictates that

$$c_0 = aS_0 + b.$$

- Substituting the previous values we obtain

$$c_0 = \frac{1}{1+R}\left(\frac{(1+R-D)-d}{u-d}c^u + \frac{u-(1+R-D)}{u-d}c^d\right).$$

- This is the same as

$$c_0 = \frac{1}{1+R}\left[qc^u + (1-q)c^d\right],$$

where $q = \frac{(1+R-D)-d}{u-d}$ defines the risk-neutral probability in the presence of a constant dividend yield D.

Numerical Example 1

- Let $S_0 = 100$, $K = 70$, $u = 1.2$, $d = 0.8$ $R = 0.04$, $D = 0.035$.
- The risk-neutral probability for this problem (with dividends) is given by

$$q = \frac{(1 + R - D) - d}{u - d} = \frac{1.01 - 0.8}{0.4} = 0.5125.$$

- Taking discounted risk–neutral expectations gives

$$c_0 = \frac{0.5125 \times 50 + 0.4875 \times 10}{1.04} = 29.3269$$

- On the other hand, the exercise value for the American call option at time zero is $S_0 - K = 30$, so that we conclude that early exercise is optimal in this case.
- Just for comparison, putting $D = 0$ gives $c_0 = 32.6923$, which confirms that optimal exercise is not optimal in the absence of dividends.

Geometric Brownian Motion with dividends

- An annualized dividend rate δ paid at every time step results in the increments of a stock price being described by

$$\Delta S \approx (\mu - \delta) S_i \Delta t + \sigma S_i \epsilon_i \sqrt{\Delta t}, \tag{1}$$

- The novelty here is that, whereas μ is the expected growth-rate of total capital invested in the stock (shares+dividends), the actual growth-rate for the stock price itself is given by $\alpha = \mu - \delta$.
- The continuous–time limit for these approximations is giving by

$$dS_t = (\mu - \delta) S_t dt + \sigma S_t dz_t. \tag{2}$$

Parameters for the binomial tree with constant dividend yield

- Suppose now that we are given a time period T (in years), the volatility σ, the mean rate of return μ and the dividend rate δ for a stock S_t following a Geometric Brownian Motion (2).
- Putting $\Delta t = T/n$, the exact same arguments used before lead us to the parameters

$$p = \frac{1 + (\mu - \delta)\Delta t - d}{u - d}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}. \quad (3)$$

- As before, the historical probability p is irrelevant for the calculations of option prices.
- Instead, we need to use the risk neutral probability

$$q = \frac{1 + (r - \delta)\Delta t - d}{u - d}.$$

Binomial tree for American calls with dividends

- Suppose we have determined the parameters $0 < d < (1 + r\Delta t) < u$ according to (3).
- To price an American call on a dividend-paying stock, we compare at each node of the tree the continuation value of the option with its exercise value.
- That is, we proceed exactly as before, but use the following expression for the value of the option at each the node

$$f^{(i,n-1)} = \max\left\{ \frac{1}{1 + r\Delta t} \left[q f^{(i,n)} + (1-q) f^{(i+1,n)} \right], K - S^{(i,n-1)} \right\},$$

where $q = \frac{1+(r-\delta)\Delta t - d}{u-d}$.

Numerical Example 2

- Let us use a binomial tree to compute the price of a 5-months American call option with strike price $K = 50$ on a stock with volatility $\sigma = 0.4$, initial price $S_0 = 50$ and annualized dividend rate $\delta = 0.04$ assuming that the annualized interest rate is $r = 0.10$.
- Let's try this on an Excel spreadsheet.