Business V703 Financial Modeling

Lecture 3: Pricing in Financial Markets

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The quest for a model for stock prices

- Recall from the last lecture that we have obtained a method to calculate the price of any European-style option using a binomial tree model.
- For practical applications, we need to choose the parameters of the tree in order to correctly approximate the behavior of a continuous time process.
- But before that, we need to come to terms with the following question: how should we model stock prices as a continuous time process ?

Assumptions for stock price behaviour

- We first make the assumption that the expected percentage return required by investors is independent of the stock price and is roughly proportional to the time period for the investment.
- That is, for short time periods, stock prices should satisfy

$$E\left[\frac{\Delta S}{S}\right] \approx \mu \Delta t, \tag{1}$$

  where μ is the expected rate of return on the stock.
- Moreover, since uncertainty grows as time goes by, we make the assumption that the variance of the percentage return is also independent of the stock price and roughly proportional to the time period for the investment.
- That is, for short time periods, we assume that

$$\mathrm{Var}\left[\frac{\Delta S}{S}\right] \approx \sigma^2 \Delta t, \tag{2}$$

  where σ^2 is called the variance of the stock.

Geometric Brownian Motion

- The most popular model for stock prices satisfying conditions (1) and (2) is the Geometric Brownian Motion model, first proposed by Samuelson in 1965.
- According to this model, the discrete–time increments of a stock price are described by

$$\Delta S \approx \mu S_i \Delta t + \sigma S_i \epsilon_i \sqrt{\Delta t}, \tag{3}$$

  where $\Delta S = S_{i+1} - S_i$, $\Delta t = t_{i+1} - t_i$ (in years) and ϵ_i denote a sequence of independent draws from a Standard Normal distribution.
- The continuous–time limit for these approximations is giving by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t, \tag{4}$$

  which should not concern us in this course.

Estimating the parameters in the model

- Let us see how the parameters μ and σ can be estimated in practice.
- We first download weekly prices for Apple shares for the past 5 months (that is, from August 17, 2006 to January 18, 2007).
- From that, we can calculate the geometric returns using the formula
$$X_{i+1} = \frac{S_{i+1} - S_i}{S_i}.$$
- Next we use a standard toolbox to calculate the sample estimates for the mean $\bar{x}$ and variance $\bar{v}$ of this time series.
- Now observe that the time interval in this example is $\Delta t = 1/52$, since there are 52 weeks in the year.
- Finally, we use (1) and (2) to obtain
$$\mu = \frac{\bar{x}}{\Delta t} \qquad \sigma = \sqrt{\frac{\bar{v}}{\Delta t}}. \tag{5}$$
- The values obtained for Apple are $\mu = 0.3599$ and $\sigma = 0.0898$.

Parameters for the binomial tree (part 1)

- Suppose now that we are given a time period T (in years), the volatility σ and the mean rate of return μ for a stock S_t following a Geometric Brownian Motion.
- We are free to choose the number of time steps n (the more time steps, the better the approximation, but also the longer the calculation gets), which fixes the time step Δt.
- All we need to do now is determine the parameters p, u and d.

Parameters for the binomial tree (part 2)

- From the expected value (1) we obtain that

$$E[S_{t+1}|S_t] = puS_t + (1-p)dS_t \approx S_t(1+\mu\Delta t). \tag{6}$$

- Next, it follows from the variance (2) that

$$\begin{aligned} \text{Var}[S_{t+1}^2|S_t] &= p(uS_t)^2 + (1-p)(dS_t)^2 - e^{2\mu}S_t^2 \\ &\approx S_t^2\sigma^2\Delta t \end{aligned} \tag{7}$$

- Finally, in order to simplify the calculations on the tree, we further impose the arbitrary condition

$$ud = 1. \tag{8}$$

Parameters for the binomial tree (part 3)

- It can be shown (you are invited to try it) that to first order in Δt, these last three equations admit the solution

$$p = \frac{1 + \mu\Delta t - d}{u - d}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}. \qquad (9)$$

- Finally, recall that the historical probability p is irrelevant for the calculations of option prices.
- Instead, we need to use the risk neutral probability

$$q = \frac{(1 + r\Delta t) - d}{u - d}.$$

Numerical Example 1

- Using the estimates for μ and σ that we found for Apple, let us calculate the binomial tree parameters for $T = 5/52 = 0.0962$, $r = 0.04$ and $n = 5$.
- We find first that $\Delta t = T/n = 1/52 = 0.0192$.
- With these, we get the dynamics parameters

$$u = e^{0.0898\sqrt{0.0192}} = 1.0125, \qquad d = 0.9876.$$

- Finally, the risk–neutral probability is given by

$$q = \frac{(1 + r\Delta t) - d}{u - d} = 0.5277$$

- Let us now calculate the price on January 19th, 2007 of a call option with strike price $K = 85$ and maturity on Feb 16, 2007.

American options

- An American call option is a contract that gives the holder the right, but not the obligation, to purchase 1 unit of the stock from the writer at any time on or before the maturity date T for a strike price K.
- An American put option is a contract that gives the holder the right, but not the obligation, to sell 1 unit of the stock to the writer at any time on or before the maturity date T for a strike price K.
- Because these options can obviously be exercised at the maturity date T, they must be at least as valuable as the corresponding European option.
- Is early exercise ever optimal ?

American calls on a non-dividend-paying stock

- At time $t < T$, consider a portfolio consisting of $Ke^{-r(T-t)}$ invested in the bank and one American call with maturity T and strike price K on a non-dividend-paying stock.
- If we were to exercise the option at any given time $t < \tau < T$ (presumably because $S_\tau > K$) then the value of the portfolio at τ would be

$$S_\tau - K + Ke^{-r(T-\tau)} < S_\tau.$$

- If we hold the option until maturity, then our portfolio will be worth

$$(S_T - K)^+ + K = \max(S_T, K) \geq S_T.$$

- Therefore, it is never optimal to exercise an American call option on a non-dividend-paying stock prior to maturity.

Economic interpretation

- The result of the previous slide is based on a typical financial math argument: the comparison between the values of two portfolios that differ only through a single decision.
- For the case of an American call on a non-dividend paying stock, we can offer the following alternative economic argument.
- If the holder of the option wants to keep the stock after time T, then it is better to wait until then and pay the strike price K (notice that this is not true if the stock pays dividends !).
- Conversely, suppose that the holder of the option thinks that the stock is overpriced (so that its price will drop soon) and wants to lock a profit of $(S_t - K)$ immediately.
- Then he is better off selling the option itself for more than $(S_t - K)$.

When is it optimal to exercise early ?

- If the stock pays dividends, it might be optimal to exercise an American call if the stock value rises sufficiently high at some time τ before maturity.
- This is because owning the stock gives the right to receive dividends, as well as benefiting from a high stock price, while owning the option only allows one to profit from the high stock price itself.
- For an American put option, however, it might be optimal to exercise early if the stock value is sufficiently low at some time τ before maturity, regardless of dividends.
- For example, if at some point the stock price is near zero, then it cannot go lower, and the holder of an American put should exercise it immediately.
- Because the binomial tree model for non-dividend-paying stocks is easier to analyze than for dividend-paying ones, we consider the example of American puts first.

Binomial tree for American puts with no dividends

- Consider again a binomial tree with parameters $0 < d < (1 + r\Delta t) < u$.
- Due to the possibility of early exercise, at each node, we need to compare the value of keeping the option alive (called the continuation value) with the amount obtained by immediate exercise (called the exercise value).
- That is, we proceed exactly as before, but use the following expression for the value of the option at each the node

$$V^{(i,n-1)} = \max\left\{\frac{1}{1+R}\left[qV^{(i,n)} + (1-q)V^{(i+1,n)}\right], K - S^{(i,n-1)}\right\},$$

where $q = \frac{(1+R)-d}{u-d}$.

Numerical Example 2

- Let us compute the price of a 5-months American put option on a 5–period binomial tree with the following parameters:

$$S_0 = 50, K = 50, u = 1.1224, d = 0.8909, r = 0.1$$

- We first construct a tree for the stock prices.
- Next we find the risk neutral probability

$$q = \frac{e^{r/12} - d}{u - d} = 0.5073.$$

- Then construct a tree for option values using the expression in the previous slide for $V^{(i,n-1)}$.
- From this we conclude that the option price at time zero is

$$P_0 = 4.48$$