Business V703 Financial Modeling

Lecture 2 - A primer in option pricing

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January 11, 2007

Market Assumptions

Let us consider a market consisting of a stock with price S_t and a bank account B_t paying a risk free interest rate r. In what follows, we make the following assumptions about this market:

- Market participants can borrow and save unlimited amounts of cash at the interest rate r.
- They can buy or sell unlimited shares at any time at the current market price.
- We can ignore trading commissions and other "market imperfections".
- No market participant can influence the price by their trades.

A simple financial derivative

- A forward contract is an agreement that obliges the holder to buy 1 unit of the stock from the writer at a specific maturity date T for a specific strike price K.
- Primary question: what is the "fair value" of this contract at an earlier time $t < T$.
- If the stock price evolves stochastically according to a probability measure P, one might be tempted to say that the fair price is given by

$$F_t = e^{-(T-t)r} E^P[S_T - K].$$

- That this value turns out to be wrong is one of the essential messages of the entire subject.

Self-Financing Replicating Portfolios

- Alternative question: can this contract be "replicated" ?
- More precisely, can we devise a trading strategy using solely the stock and the bank account which exactly reproduces the pay-off of the contract no matter what random fluctuations happen to the stock ? If so, this is called a replicating portfolio.
- In addition, we require trading strategies to be self-financing, in the sense that no additional funds are invested (or withdrawn) from the portfolio after the initial investment is made.
- **Example**: If, at time t, we buy one share and borrow $e^{-r(T-t)}K$ dollars and then do nothing but wait until time T, then we will have a self-financing portfolio worth $(S_T - K)$, which is exactly the pay-off for the forward contract.

No Arbitrage and the Law of One Price

- An arbitrage is a self-financing trading strategy which costs nothing to set up and generates non-negative cash flows with strictly positive expected value.
- A fundamental principle of financial mathematics is that arbitrages do not exist ! More mundanely, there is no such thing as a free lunch in a financial market.
- This implies the law of one price: two self-financing portfolios which generate identical (random) cash flows to the future of time t must have the same price at time t.

Pricing by replication

- The portfolio consisting of one share and $e^{r-(T-t)}K$ dollars borrowed from the bank costs

$$S_t - e^{-r(T-t)}K$$

  at time t.
- Waiting and doing nothing until time T produces a portfolio worth $(S_T - K)$.
- This is a self-financing portfolio which generates the same cash flow as the forward contract at time T.
- The law of one price then dictates that

$$F_t = S_t - e^{-r(T-t)}K$$

  is the fair value of the forward contract at time $t < T$.

Plain Vanilla Equity Options

- A European call option is a contract that gives its holder the right, but not the obligation, to buy 1 unit of the stock from the writer at a specific exercise date T for a specific strike price K.
- Its pay-off at time T can therefore be written as

$$(S_T - K)^+ := \max(S_T - K, 0).$$

- A European put option is a contract that gives its holder the right, but not the obligation, to sell 1 unit of the stock to the writer at a specific exercise date T for a specific strike price K
- Its pay-off at time T is

$$(K - S_T)^+ := \max(K - S_T, 0).$$

Dynamic hedging for options

- As before, we try to obtain the fair price of such options as the initial value of a self-financing replicating portfolio.
- Can you devise a simple portfolio (such as the one for a forward contract) for this task ?
- The problem here is that no static (i.e buy-and-hold) trading strategy can perfectly replicate calls and puts.
- Need a dynamic self-financing replicating portfolio.
- In continuous time, the mathematical apparatus for this kind of portfolio depends on Itô's formula.

The one-period binomial model

- Consider a stock with initial price S_0 at time $t = 0$ and a bank account with a simply–compounded interest rate R.
- At time $T = 1$ we assume that the price changes to

$$S_T = \begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } 1 - p, \end{cases}$$

  for some $0 < d < (1 + R) < u$.
- What is the fair price at time $t = 0$ of a call option with pay-off $(S_T - K)^+$?
- Need to construct a self-financing replicating portfolio using only the stock and the bank account.

Hedging a call in the one-period binomial model

- Suppose that, at time $t = 0$ we buy a shares and invest b dollars in the bank.
- This portfolio replicates the call option at time $T = 1$ if and only if

$$\begin{aligned} a(uS_0) + b(1+R) &= c_u := (uS_0 - K)^+ \\ a(dS_0) + b(1+R) &= c_d := (dS_0 - K)^+ \end{aligned}$$

- Solving this equations for a and b gives

$$a = \frac{c_u - c_d}{(u-d)S_0} \quad b = \frac{uc_d - dc_u}{(1+R)(u-d)}.$$

Pricing a call in the one-period binomial model

- Having found a self-financing replicating portfolio, the law of one price dictates that

$$c_0 = aS_0 + b.$$

- Substituting the previous values we obtain

$$c_0 = (1+R)\left(\frac{(1+R)-d}{u-d}c_u + \frac{u-(1+R)}{u-d}c_d\right).$$

- This is the same as

$$c_0 = \frac{1}{1+R}\left[qc_u + (1-q)c_d\right],$$

where $q = \frac{(1+R)-d}{u-d}$ defines the so called risk-neutral probability.

Numerical Example 1

- Let $S_0 = 100$, $K = 100$, $u = 1.2$, $d = 0.8$ and $R = 0.1$.
- Our previous formulas tells us that the replicating portfolio for a call option should consist of the holdings:

$$
\begin{aligned}
a &= \frac{c_u - c_d}{(u-d)S_0} = \frac{20}{40} = 0.5, \\
b &= \frac{uc_d - dc_u}{(1+R)(u-d)} = \frac{0.8 \times 20}{1.1 \times 0.4} = -36.363.
\end{aligned}
$$

- If the stock goes up, the final value for this portfolio is

  $auS_0 + b(1+R) = 0.5 \times 120 - 36.363 \times 1.1 = 20 = (uS_0 - K)^+$.

- Conversely, if the stock goes down, its final value is

  $adS_0 + b(1+R) = 0.5 \times 80 - 36.363 \times 1.1 = 0 = (dS_0 - K)^+$

- Therefore, this is a replicating portfolio, so the value for the option is

$$c_0 = aS_0 + b = 0.5 \times 100 - 36.363 = 13.637.$$

Numerical Example 1 (continued)

- Alternatively, the risk-neutral probability for this problem is given by

$$q = \frac{(1+R)-d}{u-d} = \frac{1.1-0.8}{0.4} = 0.75.$$

- Therefore, taking discounted risk–neutral expectations gives

$$c_0 = \frac{1}{1+R}\left[qc^u + (1-q)c^d\right] = \frac{0.7629 \times 20}{1.1} = 13.6364,$$

  which agrees with the previous result to significant digits.
- Notice that in this example $a > 0$, indicating that the necessary hedge for a call option requires a long position in the stock.

Numerical Example 2

Now keep the same parameters as in the previous example except:

- The initial stock price: use $S_0 = 90, 100, 110$.
- This gives $c_0 = 5.5224, 13.8060, 22.0896$, so we see that call option prices increase with S_0.
- The strike price: use $K = 90, 100, 110$.
- We now obtain $c_0 = 20.7090, 13.8060, 6.9030$, that is, call option prices decrease with K.
- The interest rate: use $r = 0.05, 0.1, 0.15$.
- For this we find $c_0 = 11.9508, 13.8060, 15.5717$, leading us to conclude that call option prices increase with the interest rate.
- The price variability: use $u = 1.15, d = 0.85$, then $u = 1.2, d = 0.8$, and finally $u = 1.3, d = 0.7$.
- This gives $c_0 = 11.5444, 13.8060, 18.3307$, from which we conclude that call option prices increase with volatility.

The multi-period model

- Consider now stock prices S_t on the dates $t = 0, 1, 2, \dots n$, given by

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p \\ dS_t & \text{with probability } 1 - p, \end{cases} \tag{1}$$

  for some initial value S_0 and parameters $0 < d < (1 + R) < u$, where R is the risk free interest rate.

- We want to price a (European) derivative whose pay-off at the terminal time $T = n$ is specified by $V_T = \Phi(S_T)$ for some function Φ (a contract).
- Need to construct a self-financing replicating portfolio using only the stock and the bank account.

Hedging in the multi-period binomial tree

- Suppose that, at the node $(i, n-1)$ on the tree, we buy $a^{(i,n-1)}$ shares and invest $b^{(i,n-1)}$ dollars in the bank.
- This portfolio replicates the random outcomes for the derivative at the adjacent nodes (i, n) and $(i+1, n)$ for the final time $T = n$ if and only if

$$
\begin{aligned}
a^{(i,n-1)}uS^{(i,n-1)} + b^{(i,n-1)}(1+R) = V^{(i,n)} \quad &:= \quad \Phi[uS^{(i,n-1)}] \\
a^{(i,n-1)}dS^{(i,n-1)} + b^{(i,n-1)}(1+R) = V^{(i+1,n)} \quad &:= \quad \Phi[dS^{(i,n-1)}].
\end{aligned}
$$

- The solution to this equations is

$$
a^{(i,n-1)} = \frac{V^{(i,n)} - V^{(i+1,n)}}{(u-d)S^{(i,n-1)}} \quad b = \frac{uV^{(i+1,n)} - dV^{(i,n)}}{(1+R)(u-d)}.
$$

- The self-financing condition is now sufficient to determine the holdings for this replicating portfolio at all nodes of the tree by solving similar equations backwards in time.

Pricing in the multi-period binomial model

- Having found a self-financing replicating portfolio, the Law of one price dictates that

$$V^{(i,n-1)} = a^{(i,n-1)} S^{(i,n-1)} + b^{(i,n-1)}.$$

- Substituting the previous values we obtain

$$V^{(i,n-1)} = \frac{1}{1+R} \left[q V^{(i,n)} + (1-q) V^{(i+1,n)} \right],$$

  where $q = \frac{(1+R)-d}{u-d}$ defines the risk-neutral probability measures.

- The value of the option in the remaining nodes is obtained by backward induction in n.

Numerical Example 3

- Let us compute the price of an European call option on a 2–period binomial tree with the following parameters:

$$S_0 = 100, K = 90, u = 1.2, d = 0.833, R = 0.1.$$

- The first step is to construct a tree for the stock prices.
- Next we find the risk neutral probability

$$q = \frac{(1+R)-d}{u-d} = 0.7275.$$

- We can then construct a tree for option values.
- From this we conclude that the option price at time zero is

$$c_0 = 26.90$$