Anticommutative algebras applied to second quantization: an example of Schönberg's formalism for Quantum Mechanics

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Abstract

In papers written in the fifties, especially in the series *Quantum Mechanics and Geometry*, Mario Schönberg introduces and analyzes several algebraic structures obtained from simple geometric objects, such as vectors and points. With these geometric algebras, he then proceedes to investigate a new algebraic formalism for the physical world, in particular for quantum mechanics, based on a deeper mathematical understanding of space-time and its relations with other properties of matter. In this work, we single out a special anticommutative algebra and, as an application to physics, relate it to the algebra of operators for fermionic second quantization.

Let E_n be an n-dimensional inner product space over a field k and let E_n^* be its dual. Denote by $B = \{\mathbf{I}_1, \ldots, \mathbf{I}_n\}$ an orthonormal basis for E_n and by $B^* = \{\mathbf{I}^1, \ldots, \mathbf{I}^n\}$ the dual basis for E_n^*, with the property that $\langle \mathbf{I}_j, \mathbf{I}^k \rangle = \delta_j^k$, where $\langle \mathbf{V}, \mathbf{U} \rangle$ denotes the linear functional $\mathbf{U}$ applied to the vector $\mathbf{V}$. The contravariant vectors in this space are then written as $\mathbf{V} = \sum_j V^j \mathbf{I}_j$, while the covariant vectors are expressed as $\mathbf{U} = \sum_j U_j \mathbf{I}^j$.

Define the associative, anticommutative, unital algebra $\mathcal{G}_n$ as the algebra generated by the elements associated with the vector $\mathbf{V}$ and $\mathbf{U}$ above satisfying the following algebraic identities:

$$[\mathbf{V}, \mathbf{V}']_+ = 0, \quad [\mathbf{U}, \mathbf{U}']_+ = 0, \quad [\mathbf{V}, \mathbf{U}]_+ = \langle \mathbf{V}, \mathbf{U} \rangle 1_{\mathcal{G}_n}, \tag{1}$$

where $[\mathbf{a}, \mathbf{b}]$ denotes the anticommutator $\mathbf{ab} + \mathbf{ba}$.

Therefore, a basis for $\mathcal{G}_n$ consists of the 2^{2n} elements of the form

$$(\mathbf{I}_1)^{r_1}\cdots(\mathbf{I}_n)^{r_n}(\mathbf{I}^n)^{s_n}\cdots(\mathbf{I}^1)^{s_1}$$

with the exponents r, s taking the values 0 and 1. That is, $\mathcal{G}_n$ is of order 2^{2n}, with a generic element written as

$$\Gamma = \sum_{p,q}^{0,\ldots,n} C^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}\mathbf{I}^{j_1}\cdots\mathbf{I}^{j_p}\mathbf{I}_{k_q}\cdots\mathbf{I}_{k_1}, \tag{2}$$

where the coefficients $C^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}$ are antisymmetric under permutations of each the sets $\{k_1,\ldots,k_q\}\subset\{1,\ldots,n\}$ and $\{j_1,\ldots,j_p\}\subset\{1,\ldots,n\}$ separately.

As it can be seen, $\mathcal{G}_n$ is the graded algebra of antisymmetric covariant and contravariant tensors of all orders, which is embedded in the usual tensor algebra.

To each element Γ above one can associate the element

$$\overline{\Gamma} = \sum_{p,q}^{0,\ldots,n} C^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}\mathbf{I}_{k_1},\cdots,\mathbf{I}_{k_q}\mathbf{I}^{j_p}\cdots\mathbf{I}^{j_1}, \tag{3}$$

which defines an invariant involution $\Gamma\mapsto\overline{\Gamma}$ in $\mathcal{G}_n$.

In the grading for $\mathcal{G}_n$ observed above, we note that the vector spaces $\mathcal{G}_{0,0}$ and $\mathcal{G}_{n,n}$, obtained respectively from the tensors with $p=q=0$ and $p=q=n$, are one dimensional, thus being isomorphic to k. In the space $\mathcal{G}_{n,n}$, let us select the elements

$$\begin{aligned}\mathbf{P} &= \mathbf{I}_1\cdots\mathbf{I}_n\mathbf{I}^n\cdots\mathbf{I}^1 &(4)\\ \overline{\mathbf{P}} &= \mathbf{I}^1\cdots\mathbf{I}^n\mathbf{I}_n\cdots\mathbf{I}_1, &(5)\end{aligned}$$

which are idempotents, that is, $\mathbf{P}^2=\mathbf{P}$ and $\overline{\mathbf{P}}^2=\overline{\mathbf{P}}$, and have the property that $\mathbf{I}_j\mathbf{P}=\mathbf{P}\mathbf{I}^j=0$ and $\mathbf{I}_j\overline{\mathbf{P}}=\overline{\mathbf{P}}\mathbf{I}_j=0$.

If we now define the pair of commuting idempotent elements $\mathbf{N}_j=\mathbf{I}^j\mathbf{I}_j$ and $\overline{\mathbf{N}}_j=\mathbf{I}_j\mathbf{I}^j$, we obtain that

$$\begin{aligned}\mathbf{P} &= \overline{\mathbf{N}}_1\cdots\overline{\mathbf{N}}_n &(6)\\ \overline{\mathbf{P}} &= \mathbf{N}_1\cdots\mathbf{N}_n, &(7)\end{aligned}$$

together with the important relation $\mathbf{N}_j+\overline{\mathbf{N}}_j=1_{\mathcal{G}_n}$.

From the elements above, we can introduce

$$\mathbf{P}^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}=\mathbf{I}^{k_q}\cdots\mathbf{I}^{k_q}\mathbf{P}\mathbf{I}_{j_p}\cdots\mathbf{I}_{j_1}, \tag{8}$$

where $j_1 < \ldots < j_p$ and $k_1 < \ldots < k_q$, with $p, q = 0, \ldots, n$. These elements are lineraly independent, due to the following multiplication rule, which follows from the properties of $\mathbf{P}$:

$$\mathbf{P}^{k_1,\ldots,k_q}_{j_1,\ldots,j_p} \mathbf{P}^{i_1,\ldots,i_r}_{h_1,\ldots,h_s} = \delta_{p,r} \delta^{i_1,\ldots,i_p}_{j_1,\ldots,j_p} \mathbf{P}^{k_1,\ldots,k_q}_{h_1,\ldots,h_s}. \tag{9}$$

Thus the set of 2^{2n} elements of the form $\mathbf{P}^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}$ constitutes a basis for $\mathcal{G}_n$, that is, we can write a generic element $\Gamma \in \mathcal{G}_n$ as

$$\Gamma = \sum_{p,q}^{0,\ldots,n} A^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} \mathbf{P}^{k_1,\ldots,k_q}_{j_1,\ldots,j_p}, \tag{10}$$

with the coefficients $A^{j_1,\ldots,j_p}_{k_1,\ldots,k_q}$ being antisymmetric as before.

The space $\mathcal{G}_n\mathbf{P}$ of antisymmetric covariant tensors of all orders can be obtained from elements of the form

$$\Psi = \sum_{p}^{0,\ldots,n} A_{j_1,\ldots,j_p} \mathbf{P}^{j_1,\ldots,j_p}. \tag{11}$$

Analogously, the space $\mathbf{P}\mathcal{G}_n$ of antisymmetric contravariant tensors of all orders can be obtained from elements of the form

$$\Phi = \sum_{p}^{0,\ldots,n} A^{j_1,\ldots,j_p} \mathbf{P}_{j_1,\ldots,j_p}. \tag{12}$$

Let us now define the *adjoint* of an element Γ as

$$\Gamma^\dagger = \sum_{p,q}^{0,\ldots,n} \left(A^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} \right)^* \mathbf{P}^{j_1,\ldots,j_p}_{k_1,\ldots,k_q}, \tag{13}$$

where A^* denotes the adjoint of the matrix A.

The map $\Gamma \mapsto \Gamma^\dagger$ is a involution which, like the map $\Gamma \mapsto \overline{\Gamma}$ introduced before, does not depend on the coordinate system used for the underlying vector space. Moreover, we have that $\Psi^\dagger = \Phi$ and $\Phi^\dagger = \Psi$. Therefore

$$\Psi^\dagger \Psi = \sum_{p}^{0,\ldots,n} \left| A_{j_1,\ldots,j_p} \right|^2 \mathbf{P} \tag{14}$$

from where we obtain a metric on antisymmetric tensors of all orders which is induced by the metric in E_n.

Returning to the involution $\Gamma \mapsto \overline{\Gamma}$, if Γ is written as in (10), then the coefficients $\overline{A}$ of $\overline{\Gamma}$ in the same basis are

$$\overline{A}^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} = (-1)^{\frac{(p-q)(p+q-1)}{2}} \delta^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} A^{j_{p+1},\ldots,j_n}_{k_{q+1},\ldots,k_n}. \tag{15}$$

From the above relation, we conclude that this involution acts on the subspace $\mathcal{G}_n\mathbf{P}$ by mapping an element of the form Ψ into one of the form $\mathbf{P}^{1,\ldots,n}\Phi$, and on the subspace $\mathbf{P}\mathcal{G}_n$ by mapping elements of the form Φ into those of the form $\Psi\mathbf{P}_{1,\ldots,n}$.

Schönberg [2, 3] observes that the application of this algebras to second quantization is carried out when we take $\mathcal{G}_n$ as an algebra over the complex numbers, which makes it the n-th order analogue of the Jordan-Wigner algebra of creation and anihilation operators for the second quantization of fermions.

If we identify the elements $\mathbf{I}_j$ and $\mathbf{I}^j$ respectively with the anihilation and creation operators, then $\mathbf{P}$ corresponds to the projection onto the vaccum state. Moreover, the elements $\mathbf{N}_j$ become identified with the number operator, while the operator for total number of particles is given by the element $\mathbf{N} = \sum_{j=1}^{n} \mathbf{N}_j$.

The state vectors for a fermionic field are given by the elements $\Psi \in \mathcal{G}_n(\mathbf{P})$, with the metric of E_n extended to it through the map $\Gamma \mapsto \Gamma^\dagger$. Finally, the map $\Gamma \mapsto \overline{\Gamma}$, which transforms $\mathbf{N}_j$ into $1_{\mathcal{G}_n} - \mathbf{N}_j$, corresponds to the permutation of particles and holes, that is, to *charge conjugation* for a fermionic quantum field.

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