Math 772 Final Project:
A Comparison of Forward Default Rates for Mean-Reverting-Jumps and CIR Affine Models

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Abstract

This is a short comparison of two models, the mean-reverting jump and CIR processes from the affine class of affine jump models in the intensity based approach to credit risk. This report is not written at the level of a paper submitted to a journal, its tone being more informal. We briefly survey for some theoretical and numerical results stemming from the work of Duffie and Singleton [1]. The field of affine jump models and their simulation is far too large to summarize in a short paper and we have opted to highlight issues regarding forward default rates and yeild spreads and the conditions that produce similar results from these different models.

Some important theoretical concepts for affine models are presented and practical results regarding simulation are demonstrated which lay the groundwork for more complex diffusions[1]. We note here that some derivations are not shown line-by-line in order to keep the presentation short, but without sacrificing the content. We chose to work in one dimensional space for the processes for simplicity and ease of presentation, extending the results to n dimensional is a staightforward exercise.

We apologize in advance for any typing errors found and non-optimal placement of figures, although the paper has been checked for typing errors, some may remain.

[1]Matlab code is available upon request

Contents

1 Introduction

Roughly stated, a jump-diffusion process X_t is termed affine if the drift, volatility and the jump arrival intensities have a constant plus linear dependence on a state random variable X_t. This defines a large class of processes that are practical for modeling and well-studied. Examples of popular affine processes are the Vasicek, Geometric Brownian Motion, and the two under study in this paper, the mean-reverting jump process (MRJ) and the CIR. The simple assumption on the diffusion coefficents being affine may limit the processes we consider to study, but this is far outweighed by the analytical tractability of the affine class of models.

The presentation of the paper starts with a very brief introduction of key results for the intensity based approach to credit risk in section 2. We introduce some basic definitions for the default intensity, survival probability and forward default rate, and set our notation. In section 3 we show how the affine stucture of the state process with an application of the Feynman-Kac formula reduces to a coupled system of two Riccati equations to determine the affine coefficients. The case for a MRJ diffusion is calculated explicitly, and the results only presented for the CIR process. In section 4 we build on the results of section 3 and derive the forward default rates for the MRJ and the CIR processes and investigate the parameter dependence of these processes. Section 5 begins a thorough comparison of the MRJ and CIR process by investigating their first and second moments. As stated in [1], a proper identification of parameters between the two models yields practically indistinguishable forward default rate curves, and here we present the main result of this paper by deriving this inter-parameter dependence by equating the moments. We also examine the free parameter depedence between the two models and its consequences for the forward default rate curves. Section 6 gives some concluding remarks about why the MRJ and CIR enjoy their special relationship within the affine framework and we also give potential pairs of candidates that may also give similar results but with considerably more effort.

2 Probability Preliminaries

It is not our intention to give a full presentation of Poisson processes used in defining the survival and default probabilities since there is a wealth of excellent references dealing with this subject[2]. We will set our notation and definitions here to hold for the remainder of the report.

We define the conditional survival probability to be the probability of survival after t years given survival up to s. Mathematically this is stated as

$$\mathbb{P}[\tau > t|\mathcal{F}_s] = \mathbb{E}_s\left[e^{\int_s^t \lambda_u du}\right] \tag{2.1}$$

where τ is the random variable

$$\tau = \inf\{t > 0 : N_t\} \tag{2.2}$$

which measures the first jump in the counting process N_t. The intensity λ_t is a process that measures the waiting time between jumps and is a compensator for the counting process N_t. That is, we define

$$\Lambda_t = \int_0^t \lambda_u du \tag{2.3}$$

such that $N_t - \Lambda_t$ is a martingale. In this report we will mostlCay be concerned with the forward default rate which is defined as

$$f(s,t) = -\partial_t \ln \mathbb{P}[\tau > t|\mathcal{F}_s] = \mathbb{E}_s\left[e^{\int_s^t \lambda_u du}\right] \tag{2.4}$$

and is an instanteous measure of the conditional survival probability at time t. The above definitions can be mapped to, or from, interest rate theory provided we identify the default intensity process λ_t with the stochastic interest rate r_t. In this fashion one may define a credit spread (yield spread) as

$$YS(s,t) = -\frac{1}{t}\partial_t \ln \mathbb{P}\left[\tau > t|\mathcal{F}_s\right] \ . \tag{2.5}$$

These above definitions will be put to use in the next sections and we now move on to introducing affine processes.

3 The Riccati Equations with Jumps

By imposing that the conditional survival probability has an exponential affine structure we introduce a substantial amount of analytical tractabililty

[2]The Math 772 course notes as an example.

but also restricts the intensity processes that one may consider. For a extensive discussion of affine processes and the limitations implied for the processes see [2]. We will consider processes of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t \, . \tag{3.6}$$

where the jump term has a Poisson counting process and a jump size distribution given by ν. We will consider the case where the jumps arrive with a Poisson intensity c and the distribution given the exponential distribution with mean jump size J[3]. More complicated jump terms may be considered and are presented in [3],[4]. In general the survival probability and generator for the above process can be written as

$$\mathbb{P}[\tau > t|\mathcal{F}_s] = \mathbb{E}_s\left[e^{\int_s^t \Lambda(X_u)du}\right] \tag{3.7}$$

and

$$\mathcal{A}f(x) = \mu(x)\partial_x f(x) + \frac{1}{2}\sigma^2(x)\partial_x^2 f(x) + \lambda(x)\int_{\mathbb{R}}\left[f(x+z) - f(x)\right]d\nu(z) \tag{3.8}$$

where λ_t is a stochatic intensity for the counting process N_t. The requirement that the functions $\mu, \sigma, \lambda, \Lambda$ are affine

$$\mu(X_t) = K_0 + K_1 X_t \tag{3.9}$$
$$\sigma(X_t) = H_0 + H_1 X_t \tag{3.10}$$
$$\lambda(X_t) = l_0 + l_1 X_t \tag{3.11}$$
$$\Lambda(X_t) = \rho_0 + \rho_1 X_t \tag{3.12}$$

will decouple the equations for the exponential affine terms in the conditional survival probability. We require that the conditional survival probability satisfies the partial differential equation via the Feynman-Kac technique

$$\mathcal{A}\mathbb{P} - \Lambda\mathbb{P} = \partial_t \mathbb{P} \tag{3.13}$$

and we subsitute the affine forms above to obtain after cancelling the common factor $\mathbb{P} = exp(\alpha + \beta\lambda)$ and arrive at the coupled set of Riccati equations

[3]This is a very simple case, but illustrates the use of affine models without clouding the presentation with complicated jump measures and Levy processes. We also note that since the jump size is exponentially distributed the jumps are positive or 'up' only.

for $\alpha(t)$ and $\beta(t)$ with the inclusion of jumps

$$\frac{d\alpha(t)}{dt} = -\rho_0 + K_0\beta(t) + \frac{1}{2}H_0(t)\beta^2(t) + l_0\left\{\theta\left[\beta(t)\right] - 1\right\} \quad (3.14)$$

$$\frac{d\beta(t)}{dt} = -\rho_1 + K_1\beta(t) + \frac{1}{2}H_1(t)\beta^2(t) + l_1\left\{\theta\left[\beta(t)\right] - 1\right\} \quad (3.15)$$

$$\theta\left[\beta(t)\right] = \int_{\mathbb{R}} e^{\beta(t)z} d\nu(z)\,, \quad (3.16)$$

where the boundary conditions are $\beta(t=0) = 0$ and $\alpha(t=0) = 0$, and the equation for the function $\theta\left[\beta(t)\right]$ can be viewed as a jump transform. Note that since the transform theta involves the measure of the jump size distribution, this integral may be difficult to evaluate analytically if the distribution is not chosen with care. For the models presented in this paper this is not the case, as we choose the jump sizes to be exponentially distributed.

$$d\nu(z) = \frac{1}{J}e^{-z/J} \quad (3.17)$$

$$\Longrightarrow \theta\left[\beta(t)\right] = \int_0^\infty e^{\beta(t)z} d\nu(z) = \int_0^\infty e^{\beta(t)z}\frac{1}{J}e^{-z/\delta} = \frac{1}{1 - J\beta(t)}\,. \quad (3.18)$$

This short calculation gives us the jump transform for the exponential jump distribution and therefore gives us the last term in the Ricatti equations for $\alpha(t)$ and $\beta(t)$

$$\theta\left[\beta(t)\right] - 1 = \frac{J\beta(t)}{1 - J\beta(t)} \quad (3.19)$$

so that the Riccati equations for exponential jumps take the form

$$\frac{d\alpha(t)}{dt} = -\rho_0 + K_0\beta(t) + \frac{1}{2}H_0(t)\beta^2(t) + \frac{l_0 J\beta(t)}{1 - \delta\beta(t)} \quad (3.20)$$

$$\frac{d\beta(t)}{dt} = -\rho_1 + K_1\beta(t) + \frac{1}{2}H_1(t)\beta^2(t) + \frac{l_1 J\beta(t)}{1 - J\beta(t)}\,. \quad (3.21)$$

3.1 Mean-Reverting Jumps

We now specialize to the case of a mean-reverting process + jumps where the jumps occur at Poisson arrival times with an intensity c and the jump sizes are exponentially distributed with a mean jump size J. The SDE for the process is

$$d\lambda_t = \kappa\left(\gamma - \lambda_t\right)dt + dJ_t \quad (3.22)$$

We can now identify the parameters in the general Ricatti equations as

$$(\rho_0, \rho_1) = (0,1) \quad (K_0, K_1) = (\kappa\gamma, -\kappa) \tag{3.23}$$

$$(H_0, H_1) = (0,0) \quad (l_0, l_1) = (0, -c) \tag{3.24}$$

and solve the following system of equations

$$\frac{d\alpha(t)}{dt} = \kappa\gamma\beta(t) - \frac{cJ\beta(t)}{1 - J\beta(t)} \tag{3.25}$$

$$\frac{d\beta(t)}{dt} = -1 - \kappa\beta(t) \, . \tag{3.26}$$

The absence of a volatility term in the SDE simplifies the generator and leaves us with a simple version of the Ricatti equations, net the jump term. The equation for $\beta(t)$ can be solved easily and gives

$$\beta_{mrj}(t) = \frac{1}{\kappa}\left(e^{-\kappa t} - 1\right) \tag{3.27}$$

while the equation for $\alpha(t)$ requires a little work. Substituting the result for $\beta(t)$ into the $\alpha(t)$ equation gives

$$\alpha_{mrj}(t) = \gamma\left(\frac{e^{-\kappa t} - 1}{\kappa} + t\right) - \frac{cJ}{J+\kappa}\int_0^t \frac{e^{-\kappa s} - 1}{1 - \frac{J}{J+\kappa}e^{-\kappa s}} ds \, . \tag{3.28}$$

But above integral can be evaluated with a simple change of variables giving the final expression

$$\alpha_{mrj}(t) = -\gamma\left(\frac{e^{-\kappa t} - 1}{\kappa} + t\right) - \frac{c}{J+c}\left\{Jt - \ln\left[1 - J\left(\frac{e^{-\kappa t} - 1}{\kappa}\right)\right]\right\} \tag{3.29}$$

or in terms of $\beta(t)$ we have

$$\alpha_{mrj}(t) = -\gamma\left(\beta_{mrj}(t) + t\right) - \frac{c}{J+c}\left\{Jt - \ln\left[1 - J\beta_{mrj}(t)\right]\right\} \tag{3.30}$$

which agrees with [1] pg.65.[4]

[4]Note to the reader, by trying to solve the Ricatti equations from the Appendix A in [1] may prove difficult. There is a negative sign error for the right-hand-sides of the equations for α and β and the correct result on pg. 65 is incompatible with this sign error. The correct signs are given here and in [5].

3.2 CIR Process

The same procedure can be done with the CIR process albeit slightly more comlicated due to the volatility term, but no jump term needs to be considered. We only quote the result

$$dX_t = \kappa\,(\theta - X_t)\,dt + \sigma\sqrt{X_t}dW_t \tag{3.31}$$

$$\alpha_{cir}(t) = \frac{2\kappa\theta}{\sigma^2}\ln\left[\frac{2\Gamma e^{(\kappa+\Gamma)t/2}}{2\Gamma + (\kappa+\Gamma)\left(e^{\Gamma t}-1\right)}\right] \tag{3.32}$$

$$\beta_{cir}(t) = \frac{-2\left(e^{\Gamma t}-1\right)}{2\Gamma + (\kappa+\Gamma)\left(e^{\Gamma t}-1\right)} \tag{3.33}$$

where

$$\Gamma^2 = \kappa^2 + 2\sigma^2\,. \tag{3.34}$$

4 Forward Default Rates and Credit Spreads

With the derivation of the solutions for $\alpha(t)$ and β in the previous section, it is a simple task to derive the forward default probability (FDR) and yield spreads (YS) for the MRJ and CIR processes. The exponential affine form is particularly suited to this caculation and is very simple due to its form. From the definition of the forward default rate

$$f(s,t) = -\partial_t \ln \mathbb{P}\left[\tau > t|\mathcal{F}_s\right] = -\partial_t e^{\dot{\alpha}(t)+\dot{\beta}(t)\lambda(s)} = -\left[\alpha(t) + \beta(t)\lambda(s)\right] \tag{4.35}$$

and the credit spread (yield spread to borrow from interest rate theory)

$$YS(s,t) = -\frac{1}{t}\partial_t \ln \mathbb{P}\left[\tau > t|\mathcal{F}_s\right] = -\frac{1}{t}\left[\dot{\alpha}(t) + \dot{\beta}\lambda(s)\right]\,. \tag{4.36}$$

To simplify matters we choose $s = 0$ and give the expressions for the FDRs and YSs for the MRJ and CIR cases which are obtained via simple differentiation w.r.t 't' designated by $\dot{(\cdot)} = \partial_t$. Below we plot some graphs in order to get a feeling for the forward default rates. Consider the CIR case first.

4.1 CIR Forward Default Rate

The expressions needed for the FDR in this case are

$$\dot{\alpha}_{cir}(t) = \frac{-2\kappa\theta(e^{\Gamma t}-1)}{[2\Gamma + (\kappa+\Gamma)(e^{\Gamma t}-1)]} \tag{4.37}$$

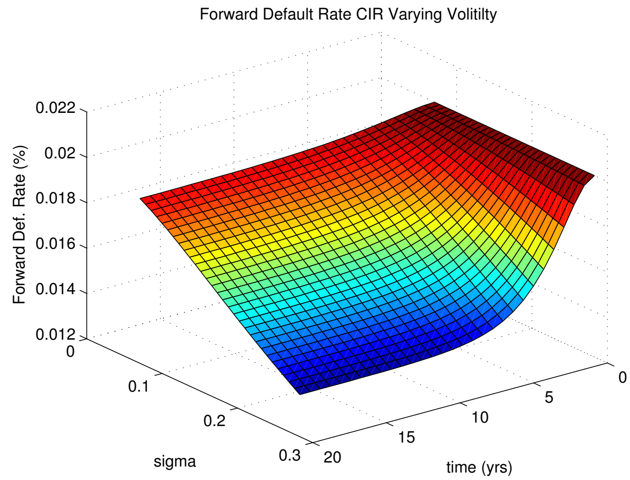

Figure 1: CIR Forward Default Rate Varying σ
Parameters: $\lambda_0 = \theta = 0.02, \kappa = 0.25$.

$$\dot{\beta}_{cir}(t) \quad = \quad \frac{-4\Gamma^2 e^{\Gamma t}}{\left[2\Gamma + (\kappa + \Gamma)(e^{\Gamma t} - 1)\right]^2} \tag{4.38}$$

and the yield spread is given as $YS_{CIR} = -(\dot{\alpha}_{CIR} + \dot{\beta}_{CIR}\lambda_0)/t$. The FDR for the CIR case is parametrized by the set $(\kappa, \Gamma, \sigma, \lambda_0)$.

We see that an increase in volatility gives a decrease in the FDR or an increase in the survival probability which can be seen as a consequence of Jensen's inequality for convex functions (probability of survival $\sim e^X$) or by power counting $\alpha \sim 1/\sigma$ which decreases for increasing σ (see Figure 1). The next figure is for varying the mean-reversion speed κ holding all other parameters fixed. We see that an increase in κ flattens out the FDR. Again by power counting we find that the FDR $\sim e^{-\sqrt{\kappa} t}$ which flattens out fast for larger κ. Both cases can be analysed by looking at the asymptotic behaviour of the FDR. A quick calculation for large t reveals that

$$FDR_{CIR} \sim \theta \left(1 - \frac{\sigma^2}{2\kappa^2}\right) \tag{4.39}$$

and we see that for increasing σ the asymptote decreases to zero, while for increasing κ the asymptote increases towards θ. The next figure is for varying the initial intensity λ_0 and we can see that for a high initial default

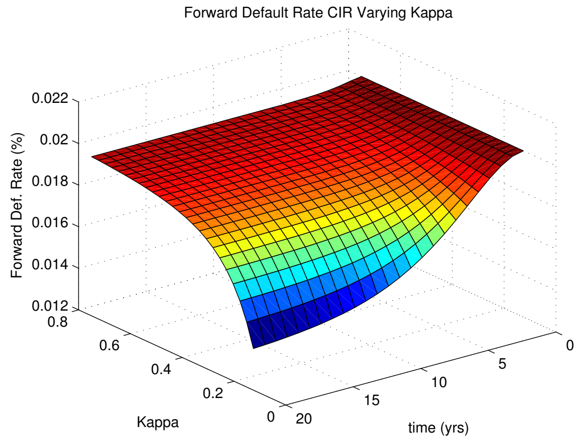

Figure 2: CIR Forward Default Rate Varying κ
Parameters: $\lambda_0 = \theta = 0.02, \sigma = \sqrt{\lambda_0} \approx 14\%$.

intensity above the asymptotic value given above, the forward default rate decreases suggesting that a firm's credit quality will improve conditional on survival. The opposite is true for firms with a low initial default rate compared to the asymptotic value. In this case there is an increase in the forward default intensity. It should be noted that the graphs of the credit spreads look very similar except for a dialation in the time axis due to the $1/t$ coefficient.

4.2 MRJ Forward Default Rate

The expressions for this case needed for the evaluation of the FDR are

$$\dot{\alpha_{mrj}} = \gamma\kappa\beta_{mrj} - \frac{cJ}{\kappa + J}\left(1 + \frac{\dot{\beta_{mrj}}}{1 - J\beta_{mrj}}\right) \tag{4.40}$$

$$\dot{\beta_{mrj}} = -e^{-\kappa t} = -\left(\kappa\beta mrj + 1\right) \tag{4.41}$$

and we observe that the FDR is parametrized by the set $(\kappa, \gamma, c, J, \lambda_0)$. Below we give graphs to investigate the role of the parameters. As in the case it is helpful to find the asymptotic behaviour of the FDR

$$\text{large t} \quad FDR_{mrj} \sim \gamma + \frac{cJ}{\kappa + J} = FDR_\infty \tag{4.42}$$

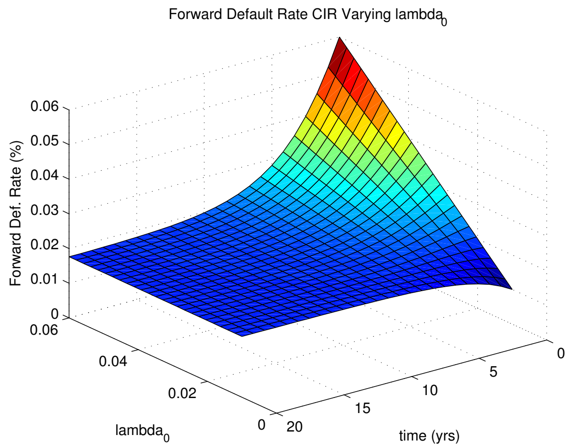

Figure 3: CIR Forward Default Rate Varying λ_0
Parameters: $\kappa = 0.25, \theta = 0.02, \sigma = \sqrt{\theta} \approx 14\%$.

which is similar the long-run mean of the MRJ process. We see that increasing κ lowers the long-run mean as well as increasing the convexity (if $\lambda_0 > FDR_\infty$) and concavity (if $\lambda_0 < FDR_\infty$) of the FDR curve (see Figure 4). The γ parameter is somewhat boring as it has a linear change in the FDR_∞ level of the FDR curve and a slight enhancement to the convexity/concavity as it decreases (see Figure 5). More intersting is what happens when we increase the jump intensity c from zero (no jumps) to $c = 1$ (a mean jump arrival rate of once per year, see Figure 6). For large initial default intensites (say $\lambda = 0.08$) and large jump intensities we see a hump develop in the FDR curve and for large times the curve flattens out above the initial default intensity. This implies that a firm's credit quality will degrade and then improve slightly over the long run. In the same case but with low jump intensities we see that the FDR curves are convex suggesting an imporvement in a firm's credit quality. If the initial default intensity is low (say $\lambda = 0.01$), the hump in the curve eventually appears at later times for all values of c, indicating an inevitable degradation in the firm's credit quality. If we now consider the case of varying the jump sizes we see that for large initial default intensities the FDR decays to the FDR_∞ for all J, but for small initial default intensities, it is the opposite, we see an increase in the FDR to FDR_∞ for all J (see Figures 7,8).

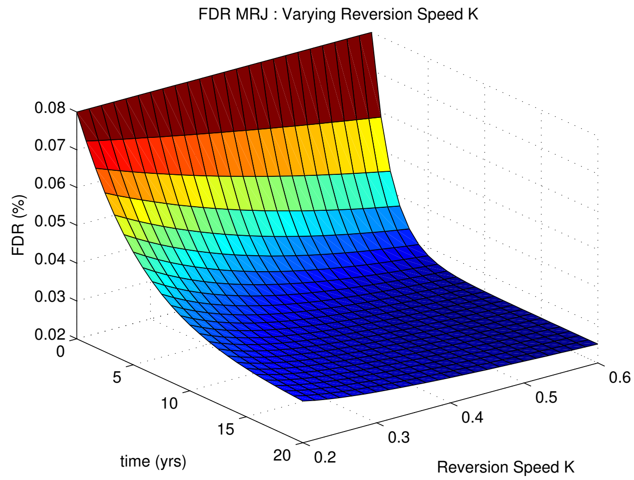

Figure 4: MRJ Forward Default Rate Varying κ
Parameters: $c = 0.02, J = 0.2, \gamma = 0.02, , \lambda_0 = 0.08$.

Although the both the CIR and MRJ model are simple in form, they posses a certain degree of complexity with regards to their parameter spaces and some similarities in their FDR curves. We now turn to investigating the relationship between the two models and what this relationship implies for the FDR curves.

5 Moments of the MRJ and CIR Processes

Here we present how the two different process, the MRJ and CIR, can be related by investigating the mean and variance and making a suitable choice of parameters. With a proper choice of interparameter dependence we look at the implications for the FDR curves for both the MRJ and CIR. We also attempt to find similar relationship between other pairs of processes, and investigate what conditions must be satisfied in order for the parameter identification via moment matching technique to work.

5.1 Moment Matching

To arrive at the conditional moments for a diffusion process there are two equally valid methods. One can take derivatives of the characteristic func-

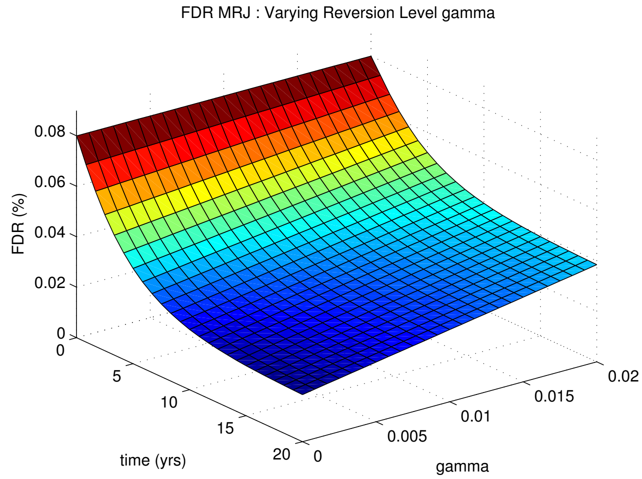

Figure 5: MRJ Forward Default Rate Varying γ
Parameters: $c = 0.02, J = 0.2, \kappa = 0.25, \lambda_0 = 0.08$

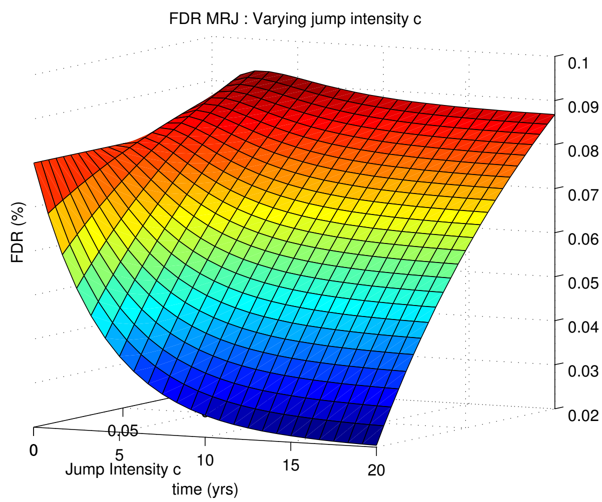

Figure 6: MRJ Forward Default Rate Varying c
Parameters: $J = 0.2, \kappa = 0.25, \gamma = 0.02, \lambda_0 = 0.08$.

tion or by taking expectations of the process itself. Although the characteristic funcition is mathematically elegent in some cases it may be difficult to

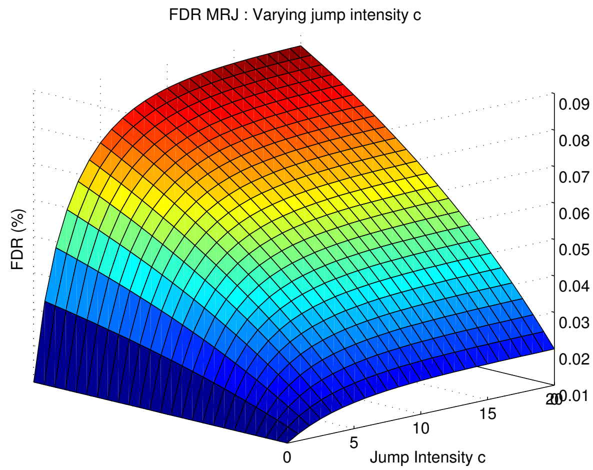

Figure 7: MRJ Forward Default Rate Varying c
Parameters: $J = 0.2, \kappa = 0.25, \gamma = 0.02, \lambda_0 = 0.01$.

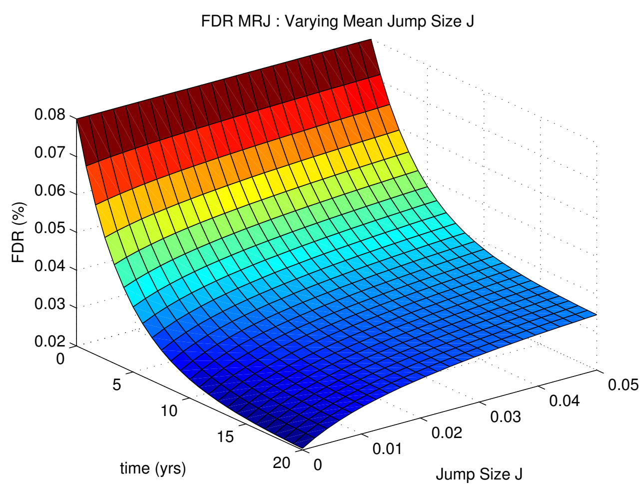

Figure 8: MRJ Forward Default Rate Varying J
Parameters: $c = 0.02, \kappa = 0.25, \gamma = 0.02, \lambda_0 = 0.08$.

obtain[5]. Taking expectations is almost always straightforward but leads to a

[5] For examples on calculating the characteristic function see [6]

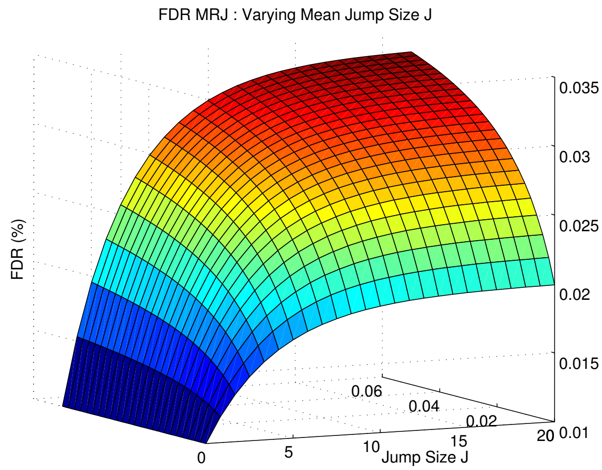

Figure 9: MRJ Forward Default Rate Varying J
Parameters: $c = 0.02, \kappa = 0.25, \gamma = 0.02, \lambda_0 = 0.01$.

matrix of ODEs that must be solved recursively for higher order conditional moments [7],[8][6] . In the case of the MRJ and CIR we opted for the second approach and present it below.

Here we outline a method of obtaining the conditional moments for a general class of jump-diffusion process. Consider the process jump-diffusion process

$$d\lambda_t = \mu(\lambda_t)dt + \sigma(\lambda_t)dW_t + J_t dN(\rho_t t) \qquad (5.43)$$

where W_t is standard Brownian motion, J_t is the jump size distribution with density function $\nu(J)$, and $N(\rho_t t)$ is a Poisson driving process with intensity fucntion ρ_t, and the jumps are independent of the Brownian term. To derive the conditional moments for all orders up to including the k^{th} order for the above diffusion one employs the generalized Ito formula for jumps on powers of λ_t through to λ_t^k and then taking expectations. The result is

$$\mathbb{E}_s(\lambda_t^k) \quad = \quad \lambda_t^k + \mathbb{E}_s\left[\int_s^t \left(k\mu(\lambda_u)\lambda_u^{k-1} + \frac{1}{2}k(k-1)\sigma^2(\lambda_u)\lambda_u^{k-2}\right) du\right]$$

[6]There is an error in this paper regarding the system of ODEs for the jump case. The correct version is presented here

$$+\mathbb{E}_s\left[\int_s^t \rho_u \mathbb{E}_J\left[(\lambda_u + J_u)^k - \lambda_u^k\right] du\right].$$

Assuming we have enough regularity to interchange the order of integration with the expectation , and then take derivatives w.r.t t we obtain the system of non-linear ODEs

$$\frac{d\mathbb{E}_s(\lambda_t^k)}{dt} = \mathbb{E}_s\left[k\mu(\lambda_u)\lambda_u^{k-1} + \frac{1}{2}k(k-1)\sigma^2(\lambda_u)\lambda_u^{k-2} + \rho_u\mathbb{E}_J\left[(\lambda_u + J_u)^k - \lambda_u^k\right]\right] \tag{5.44}$$

with boundary conditions $\mathbb{E}_s(\lambda_s^k) = \lambda_s^k$. The above formula for the conditional moments is extremely useful in general, and takes on even more appeal if the processes under study have affine coefficients. Both the MRJ and the CIR have affine coefficients and we now turn to the evaluation of their first and second moments. In the following calculations we will take $s = 0$ and $\mathbb{E}_0(\rho_t) = c$, a constant intensity from the exponential distribution, and J_t independent of time for simplicity.

5.2 Moments of CIR+Jumps

Here we show how the above method can be applied to a CIR + jumps model (CIRJ)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma_c(\lambda_t)dW_t + J_t dN(\rho_t t)\,. \tag{5.45}$$

The regular CIR is retrieved in the limit that the jumps go to zero ($c_c \to 0$) and the MRJ by setting the volatility to zero ($\sigma_c \to 0$). To obtain the first and second moments we use $k = 1, 2$ in the above to obtain the system

$$\dot{m}_1 = -\kappa m_1 + \kappa\theta + c\mathbb{E}(J_c) \tag{5.46}$$

$$\dot{m}_2 = -2\kappa m_2 + \left[2\kappa\theta + \sigma_c^2 + 2c_c\mathbb{E}(J_c)\right] + c_c\mathbb{E}(J_c^2) \tag{5.47}$$

where $m_1 = \mathbb{E}_t(\lambda_t), m_2 = \mathbb{E}_t(\lambda_t^2)$ are the first and second moments, c_c is the intensity of the jumps for the CIRJ process, and J_c is the jumps in CIRJ, the density is general. The above system of equations can easily be solved with the use of intergrating factors, and we pass over the derivation and only quote the result

$$\mathbb{E}_0(\lambda_t) = \lambda_0 e^{-\kappa t} + \bar{\theta}\left(1 - e^{-\kappa t}\right) \tag{5.48}$$

$$
\begin{aligned}
\mathbb{E}_0(\lambda_t^2) &= \lambda_0^2 e^{-2\kappa t} + \frac{c_c \mathbb{E}(J_c^2)}{2\kappa} && (5.49)\\
&+ 2\left(\bar{\theta} + \frac{\sigma_c^2}{2\kappa^2}\right)\left[\lambda_0\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\bar{\theta}}{2}\left(1 - e^{-\kappa t}\right)^2\right]\\
\Longrightarrow Var(\lambda_t) &= \frac{\lambda_0 \sigma_c^2}{\kappa} e^{-\kappa t}\left(1 - e^{-\kappa t}\right) + \frac{\sigma_c^2 \bar{\theta}}{2\kappa}\left(1 - e^{-\kappa t}\right)^2 && (5.50)\\
&+ \frac{c_c \mathbb{E}(J_c^2)}{2\kappa}\left(1 - e^{-2\kappa t}\right)
\end{aligned}
$$

where $\bar{\theta} = \theta + c_c \mathbb{E}(J_c^2)/\kappa$. This is just the usual CIR mean and variance but with $\theta \to \bar{\theta}$ and an additional term due to the variance of the jumps. The large t limit gives

$$
\begin{aligned}
t \to \infty \quad \mathbb{E}_0(\lambda_t) &= \theta + \frac{c_c \mathbb{E}(J_c)}{\kappa} && (5.51)\\
t \to \infty \quad Var(\lambda_t) &= \frac{\sigma_c^2 \theta}{2\kappa} + \frac{c_c}{2\kappa}\left(\frac{\sigma_c^2}{\kappa}\mathbb{E}(J_c) + \mathbb{E}(J_c^2)\right) && (5.52)
\end{aligned}
$$

so that the mean and variance of the CIRJ are enhanced by the mean and variance of the jumps relative to the regular CIR process.

5.3 Moments of MRJ + Volatility

If we add a Brownian term $\sigma_m \sqrt{\lambda_t} dW_t$ to the MRJ process the new process is the same as the CIR considered above but with different parameters for the volatility σ_m, the mean-reversion level γ, and jump parameters c_m, J_m. We could have added a constant or geometric Brownian term, but we would like to keep the process positive since we will make a comparision to the CIR process. Although we have imposed a restriction on the Brownian term, we have not imposed the same jump size distribution, this is free to choose as long as the jumps are positive. The form of the results apply here and we give the large t form for the mean and variance

$$
\begin{aligned}
t \to \infty \quad \mathbb{E}_0(\lambda_t) &= \gamma + \frac{c_m \mathbb{E}(J_m)}{\kappa} && (5.53)\\
t \to \infty \quad Var(\lambda_t) &= \frac{\sigma_m^2 \gamma}{2\kappa} + \frac{c_m}{2\kappa}\left(\frac{\sigma_m^2}{\kappa}\mathbb{E}(J_m) + \mathbb{E}(J_m^2)\right)\ . && (5.54)
\end{aligned}
$$

5.4 Moment Matching MRJ and CIR

By ignoring the higher order moments which generate skewness and kurtosis we can make a rough parameter equivalence bewteen the two processes provided we have enough initial information regarding their jump processes. As simple case, suggested by Duffie [1] is to consider a MRJ process with exponentially distributed jumps and compare it to a CIR. Equating the first and second moments above and setting $\sigma_m = c_c = 0$ and $\sigma_c, = \sigma, c_m = c, J_m = J$ immediately gives

$$\gamma = \theta - \frac{cJ}{\kappa} \quad (5.55)$$

$$J = \sqrt{\frac{\sigma^2\theta}{2c}}\,. \quad (5.56)$$

If we are given the intensity c then we have completely specified the MRJ parameters in terms of the CIR parameters which we assume are known, and recall that κ is the same for both processes. We now investigate what this parameter identification implies for the forward default rates for both MRJ and CIR.

6 Moment Matching and the Forward Default Rate

In section 3.5.3 of Duffie's book 'Credit Risk', the claim is made that after choosing the jump parameters c and J of MRJ to match the moments of λ_t for CIR there is little difference in the forward default rate curves. This paper has been building to prove this claim in the affirmative and we are in a position to give quantitative support. From section 4, we have definite values for the CIR parameters

$$\kappa = 0.25, \quad \theta = 0.02, \quad \sigma = \sqrt{\theta}, \quad \lambda_0 = \text{free parameter} \quad (6.57)$$

and for the MRJ we have the intensity $c = 0.02$ which gives the moment matching conditions as

$$J = \sqrt{\frac{\theta\sigma^2}{2c}} = 0.1 \quad \text{and} \quad \gamma = \theta - \frac{cJ}{\kappa} = 0.012\,. \quad (6.58)$$

Below are plots which give the forward default rate and credit spread for different λ_0 showing an exact agreement with the graphs in [1]. The two processes are pratically identical for small values of λ_0 deviating less than 5 basis points over the 20 year time frame and for larger values, say $\lambda_0 = 0.06$, the deviation is less than 20 basis points over the same time period. A similar statement can be said about the credit spread as well (see Figures

10-13). Differences to start to appear once we change the parameters c, J, κ. Let us first consider the role that κ has in altering the near-perfect fit. As stated previously, increasing κ is equivalent to increasing the FDR_∞ of both processes. We provide plots of the FDR curves for $\kappa = 0.1, 0.15, 0.25, 0.5$ for a $\lambda_0 = 0.3$ and $c = 0.02, J = 0.1$ and note the increase in the long-run mean (see Figures 14-17). As well, the striking feature in this set of graphs is the hump shaped and decaying FDR for the MRJ model under the particular choice of of $\kappa <\approx 0.1$ (verifed by simulation). In this case there is also a large deviation between the two curves. As we increase κ this deviation lessens to a negligable amount at $\kappa = 0.5$.

This behaviour sprurred the question of whether the particular choice of the parameters $c = 0.02$ and $\kappa = 0.25$ used for deriving the similarities of the MRJ and CIR forward default rate curves was somehow chosen by luck or carefully selected to achieve a positive result. Rather than trying to optimize all three parameters c, J, κ simultaneously we chose a quick first-order approximation by optimizing to each one seperatly while holding the other two fixed at their given values. We calcuated the RMSE between the MRJ and CIR forward default rate curves for different $\lambda_0 \in [0, 0.1]$ with a $\Delta\lambda_0 = 0.002$ as we iterated through the parameter space under consideration, and averaged in the 50 year period. Below are three plots (Figures 18-20) which show that under this approximation to the full simultaneous optimization of all three parameters, we obtain $c = 0.0225, J = 0.0173, \kappa = 0.2490$, which compares well with the values of $c = 0.02, J = 0.1, \kappa = 0.25$. Since the RMSE is convex for each case and has an quadratic-type dependence we could be almost safe to say that this is a stable point in the parameter space. It would seem that this choice of parameters in [1] was serendipity.

7 Conclusions

The choice of a mean-reverting jump model with exponential jump sizes and jump intensity is possibly the simplest case to consider when comparing it to a CIR process. Both are members of the affine class and have closed form solutions for the survival probabilities, forward default rates and credit spreads which we have shown explicitly. Since both are strickly positive and mean-reverting we expect some similarities to appear after a analysis of the moments. We may try to consider more complex diffusions with the addition of a Brownian term, but this would introduce complications on two

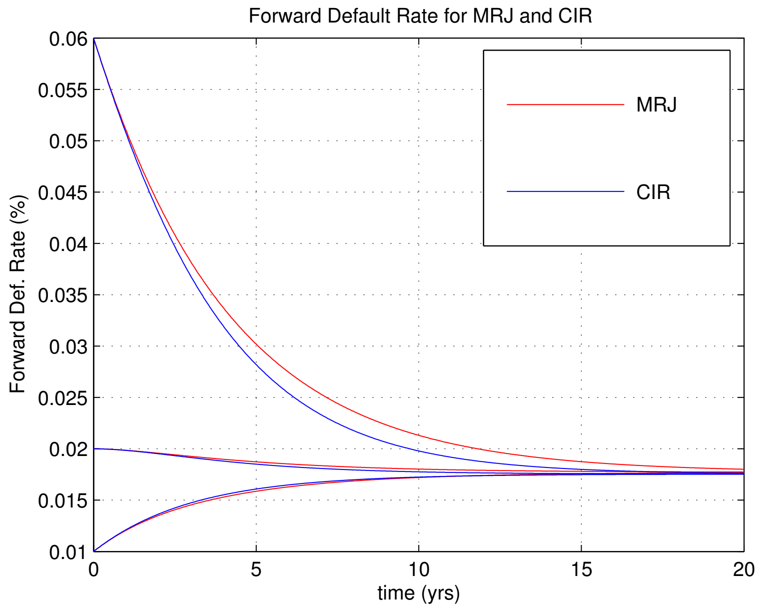

Figure 10: FDR for MRJ and CIR
Parameters:$\lambda_0 = 0.01, 0.02, 0.06$

levels. The diffusion may go to negative values which is an unattractive feature for modeling intensity processes and leads to non-sensical probabilities and secondly, the procedure of moment matching would only become more complicated with another parameter to match. Another possbile extension is to change the jump distribution for only positive jumps. Potential candidates could be the $Gamma(\alpha, \beta)$ or $Weibull(\alpha, \beta)$ distributions. Although the moment matching equations were derived for general distributions, the drawback in using more complicated jump distributions is that the variance of the distribution would have to be matched. This is incompatible with a straight CIR process, but may be possible if a positive jump term is added (CIRJ). In addition we have shown that for the MRJ and the CIR a particular parameter identification leads to almost equivalent forward default rates. The choice of remaining parameters, in particular the mean-reversion rate κ does not uphold this property uniformly in its parameter space. A numerical investigation showed that the choice $\kappa = 0.25$ used initially is an optimal point.

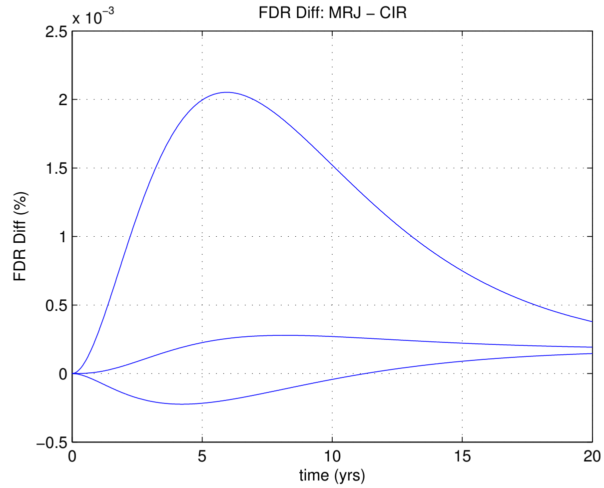

Figure 11: FDR difference MRJ-CIR
Parameters:$\lambda_0 = 0.01, 0.02, 0.06$

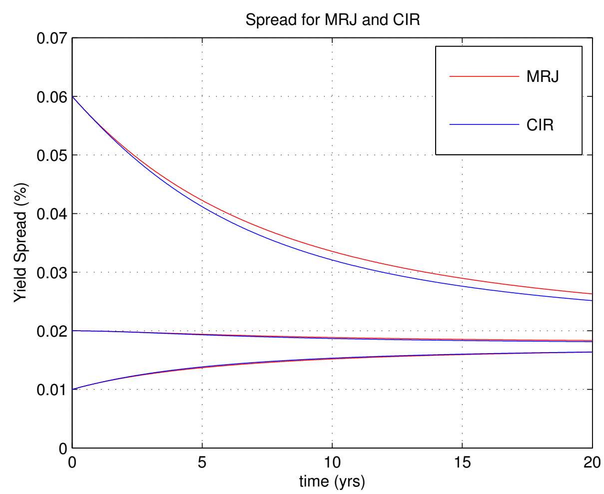

Figure 12: Spread for MRJ and CIR
Parameters:$\lambda_0 = 0.01, 0.02, 0.06$

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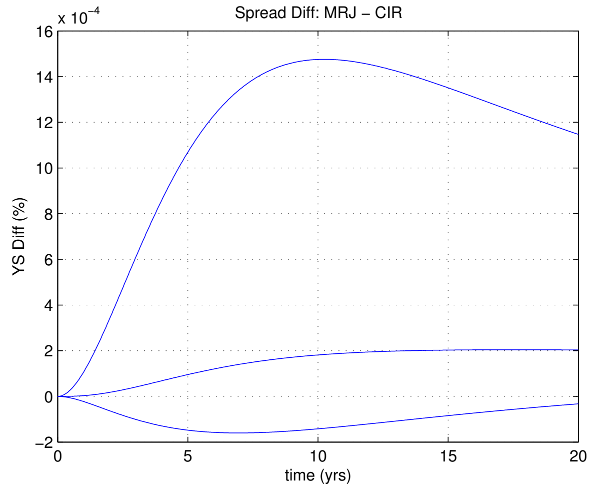

Figure 13: Spread difference MRJ-CIR
Parameters:$\lambda_0 = 0.01, 0.02, 0.06$

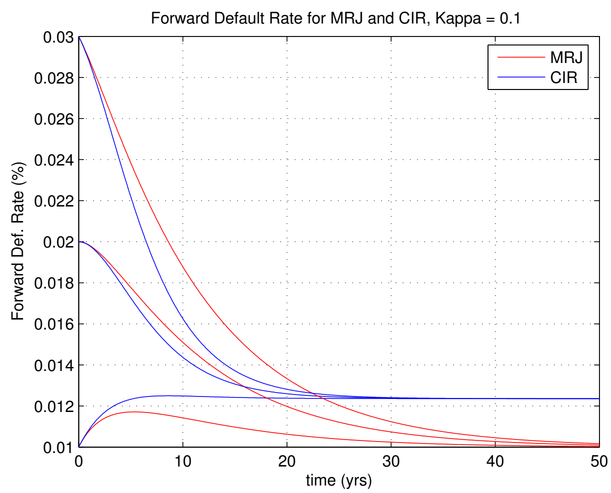

Figure 14: FDR MRJ and CIR $\kappa = 0.1$
Parameters:$c = 0.02, J = 1, \lambda_0 = 0.01, 0.02, 0.03$

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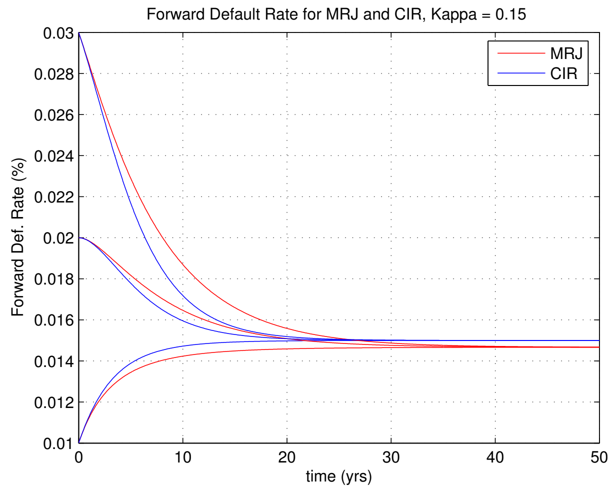

Figure 15: FDR MRJ and CIR $\kappa = 0.15$
Parameters:$c = 0.02, J = 1, \lambda_0 = 0.01, 0.02, 0.03$

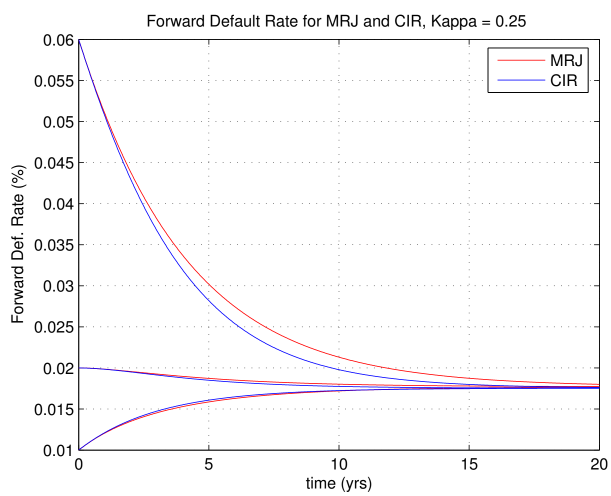

Figure 16: FDR MRJ and CIR $\kappa = 0.25$
Parameters:$c = 0.02, J = 1, \lambda_0 = 0.01, 0.02, 0.03$

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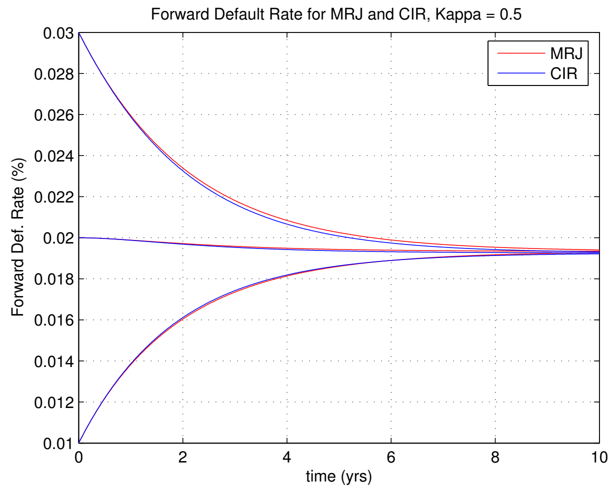

Figure 17: FDR MRJ and CIR $\kappa = 0.5$
Parameters:$c = 0.02, J = 1, \lambda_0 = 0.01, 0.02, 0.03$

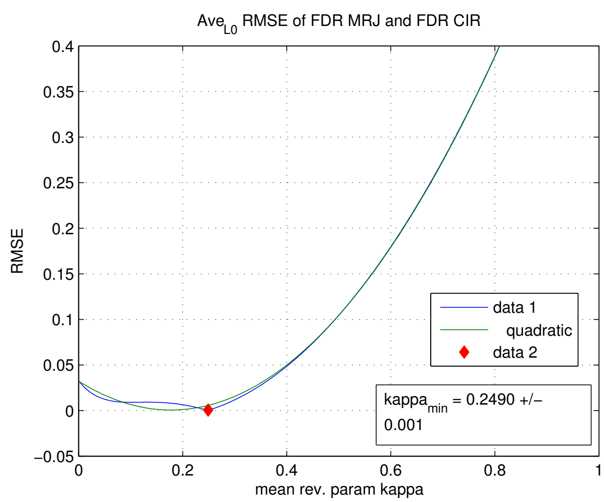

Figure 18: $< RMSE >_{\lambda_0} \quad \lambda_0 \in [0, 0.1]$
Parameters:$\Delta\lambda_0 = 0.002, J = 0.1, c = 0.02, T = 50yrs$

1343-1376 (2000).

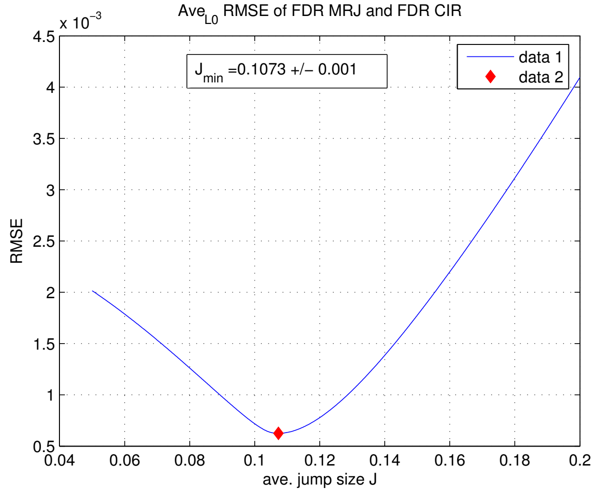

Figure 19: $< RMSE >_{\lambda_0} \quad \lambda_0 \in [0, 0.1]$
Parameters:$\Delta\lambda_0 = 0.002, \kappa = 0.25, c = 0.02, T = 50yrs$

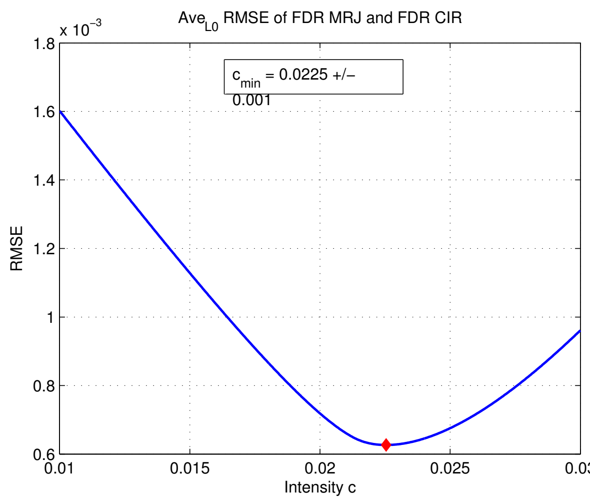

Figure 20: $< RMSE >_{\lambda_0} \quad \lambda_0 \in [0, 0.1]$
Parameters:$\Delta\lambda_0 = 0.002, \kappa = 0.25, J = 0.1, T = 50yrs$

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