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Getting Real with Real Options: A Utility–Based Approach for Finite–Time Investment in Incomplete Markets

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Abstract: We apply a utility–based method to obtain the value of a finite–time investment opportunity when the underlying real asset is not perfectly correlated to a traded financial asset. Using the comparison principle for the associated variational inequality, we establish several qualitative properties of the optimal investment boundary, in particular its dependence on correlation and risk aversion. We then use a discrete–time algorithm to calculate the indifference value for this type of real option and present numerical examples for the corresponding investment thresholds. We verify that even in the zero correlation case, whereby none of the risk in the project can be hedged in a financial market, the paradigm of real options can still be applied to value an investment decision, since the opportunity to invest still carries an option value above its net present value. In other words, it is time flexibility itself, more than the possibility of replication, that is the source of the extra value of an investment opportunity. This value, however, quickly erodes at higher levels of risk aversion, and even more so when the project is weakly correlated to financial markets.

Keywords: real options, incomplete markets, exponential utility, optimal exercise policy

1. INTRODUCTION

Most of the standard literature in real options is based on one or both of the following unrealistic assumptions: (i) that the time horizon for the problem at hand is infinite and (ii) that the real asset under consideration is perfectly correlated to a traded financial asset. The *infinite–maturity* hypothesis helps to reduce the dimensionality of the problem by removing its dependence on time, therefore allowing concentration on stationary solutions only. The *spanning–asset* hypothesis allows the introduction of useful replication arguments developed for derivative pricing in complete markets. Together they led to the development of a coherent and intuitive approach for

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investment under uncertainty, well–represented for instance in Dixit and Pindyck (1994).

Since then, several authors have dropped the artifice of an infinite time horizon and used standard numerical methods to deal with the corresponding non-stationary valuation problem. These include finite–difference methods for the associated partial differential equation and lattice methods for discrete–time option pricing. Even closed-form solutions for certain types of options can still be obtained in the finite–maturity case as long as one is prepared to work within the Black–Scholes framework, as is done for example in Shackleton and Wojakowski (2007). In other words, removing the first unrealistic assumption above appears to be a minor problem provided the second unrealistic assumption is maintained.

In this regard, even recent books such as Smit and Trigeorgis (2004) carry the assumption that 'real-options valuation is still applicable provided we can find a reliable estimate for the market value of the asset' (p. 102), which is tantamount to saying that 'markets are sufficiently complete'. In reality, most investment problems where the real options approach is deemed relevant occur in markets which are far from being complete. For example, almost by definition an R&D investment decision concerns a product which is *not* currently commercialized and therefore commands uncertainty that is at best imperfectly correlated with available financial assets.

Exceptions to this adherence to a 'near completion' assumption, but still in the context of an infinite time horizon, are Hugonnier and Morellec (2007) and Henderson (2007). In the first paper, a risk averse manager facing an investment decision tries to maximize his expected utility considering the effect that shareholders' external control will have on his personal wealth. By assuming that the underlying project is subject to both market risk, which the manager can hedge using a traded financial asset, and idiosyncratic risk, which cannot be hedged in the available financial market, the authors reduce this decision to an optimal portfolio problem in an incomplete market. Under a similar model for a project with both market and idiosyncratic risks, Henderson (2007) uses an exponential utility framework in order to actually calculate the value for the investment opportunity as a derivative in an incomplete market, therefore remaining closer in spirit to the real options paradigm.

In this paper, we study a finite–horizon version of Henderson's model. We first review the mechanism for pricing American–style derivatives in incomplete markets using an exponential utility in Section 2*(i)*, followed by its formulation as a free–boundary problem in Section 2*(ii)*. We use the comparison principle for variational inequalities to establish several properties of the optimal exercise boundary in Section 2*(iii)*. For actual computations, we propose a binomial approximation in Section 3. Such approximation is similar to the binomial model proposed in Detemple and Sundaresan (1999), except that our use of an exponential utility function renders a much smaller computational burden, leading to a computational complexity identical to that of a standard Cox–Ross–Rubenstein tree (see Cox et al., 1979). This is followed by numerical experiments exploring the properties of the option to invest in Section 4, including comparisons with the corresponding infinite horizon and complete market limits. Section 5 then presents conclusions drawn from the model, especially in contrast with alternative ways of dealing with market incompleteness in the context of real options.

2. THE CONTINUOUS–TIME MODEL

(i) The Indifference Value

Consider an investor contemplating the decision to invest in a project with value given by a positive stochastic process $\overline{V}_t$ by paying a sunk cost $I(t) = e^{\alpha(t-t_0)}I$ with initial value $I(t_0) = I$ and growing in time at a rate $\alpha \geq 0$. Assuming that the investment decision can be made at any time $t_0 \leq t \leq T \leq \infty$, the opportunity to invest is formally equivalent to an American call option with strike price $I(t)$ having the project value as the underlying asset. When the project value is perfectly correlated to the price of a traded financial asset, such option can be priced using standard arbitrage and replication arguments, following the pioneering approach of Brennan and Schwartz (1985). In the absence of such spanning asset, the option to invest becomes analogous to a derivative in an incomplete market. Instead of wishfully pretending that risk–neutral and replication arguments can still be used in this case, we argue that the investor's risk preference should be explicitly used for valuing the option to invest. For this, we follow Henderson and Hobson (2002) and consider a utility indifference framework based on an exponential utility of the form $U(x) = -e^{-\gamma x}$, where $\gamma > 0$ is the risk–aversion coefficient.

We consider a liquidly traded financial asset whose price $\overline{S}_t$ is partially correlated to the value of the project $\overline{V}_t$, as well as a bank account with normalized value $B_t = e^{r(t-t_0)}$ for a constant interest rate $r \geq 0$. As it is common in the optimal investment literature, we take into account the time–value of money by doing the analysis in terms of the *discounted* values $S_t = \overline{S}_t/B_t$ and $V_t = \overline{V}_t/B_t$, which are henceforth assumed to satisfy:

$$
\begin{aligned}
dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW_t^1 \\
dV_t &= (\mu_2 - r)V_t dt + \sigma_2 V_t \big(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2\big),
\end{aligned}
\tag{1}
$$

for $t_0 \leq t \leq T \leq \infty$, where $W = (W^1, W^2)$ is a standard two–dimensional Brownian motion. Here $\mu_1, \mu_2 \in \mathbb{R}$ are the expected growth rates for the financial asset $\overline{S}_t$ and the project value $\overline{V}_t$, $\sigma_1, \sigma_2 > 0$ are their volatilities, and $-1 < \rho < 1$ is their instantaneous correlation.

We suppose further that the investor trades dynamically in the financial market by holding H_t units of the asset with discounted price S_t and investing the remainder of his wealth in the bank account. Denoting the discounted amount of money invested in the financial asset by $\pi_t = H_t S_t$, it follows that the discounted value of the corresponding self–financing portfolio satisfies:

$$
X_t^\pi = \int_0^t H_s\, dS_s = \int_0^t \pi_s \frac{dS_s}{S_s},
\tag{2}
$$

which can be expressed in differential form as:

$$
dX_t^\pi = \pi_t(\mu_1 - r)\, dt + \pi_t \sigma_1 dW_t^1, \quad t_0 \leq t \leq T.
\tag{3}
$$

In the absence of any investment opportunity in the project V_t, the optimal investment in the financial asset S_t is described by the Merton value function (see Merton, 1969):

$$M(t, x) = \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[-e^{-\gamma X_T^\pi} \big| X_t^\pi = x\right] = -e^{-\gamma x} e^{-\frac{(\mu_1 - r)^2}{2\sigma^2}(T-t)}, \tag{4}$$

for $t_0 \leq t \leq T$, where X_t^π follows the dynamics (3) and $\mathcal{A}_{[t,T]}$ is the set of admissible investment policies on the interval $[t, T]$, which we take to be progressively measurable processes satisfying the integrability condition $\mathbb{E}[\int_t^T \pi_s^2 ds] < \infty$.

As mentioned before, we model the opportunity to invest in the project as a decision to pay an amount $I(t)$ in return of uncertain future cash flows whose discounted market value at time t is given by V_t. In other words, the decision to invest in the project at a random time τ corresponds to a discounted payoff:

$$C_\tau = (V_\tau - e^{(\alpha - r)(\tau - t_0)} I)^+,$$

which we recognize as the formal analogue of an American call option. To value this option we make the assumption that, having exercised the option at time τ, the investor adds its discounted payoff to his discounted wealth X_τ^π at time τ and then continues to invest optimally until time T. Accordingly, the investor needs to solve the following optimization problem:

$$u(t_0, x, v) = \sup_{\tau \in \mathcal{T}[t_0, T]} \sup_{\pi \in \mathcal{A}_{[t,\tau]}} \mathbb{E}\left[M(\tau, X_\tau^\pi + C_\tau) \big| X_{t_0}^\pi = x, V_{t_0} = v\right], \tag{5}$$

where $\mathcal{T}[t_0, T]$ denotes the set of stopping times in the interval $[t_0, T]$. Following Hodges and Neuberger (1989), we define the indifference value for the option to invest as the amount p satisfying:

$$M(t_0, x) = u(t_0, x - p, v). \tag{6}$$

That is, p is the amount of money that the investor is prepared to spend at time t_0 in order to acquire this option. For instance, p might be the price of land that will allow a subsequent real estate development, or the price of a one–off license to explore a natural resource.

(ii) The Free–Boundary Problem

It follows from the dynamic programming principle that the value function u is the solution to the free boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_\pi \mathcal{L}^\pi u \leq 0, \\[2mm] u(t, x, v) \geq \Lambda(t, x, v), \\[2mm] \left(\dfrac{\partial u}{\partial t} + \sup_\pi \mathcal{L}^\pi u\right) \cdot (u - \Lambda) = 0, \end{cases} \tag{7}$$

where:

$$\mathcal{L}^\pi = (\mu_2 - r) v \frac{\partial}{\partial v} + \frac{\sigma_2^2 v^2}{2} \frac{\partial^2}{\partial v^2} + \pi(\mu_1 - r) \frac{\partial}{\partial x} + \rho \pi \sigma_1 \sigma_2 v \frac{\partial^2}{\partial x \partial v} + \frac{\pi^2 \sigma_1^2}{2} \frac{\partial^2}{\partial x^2} \tag{8}$$

is the infinitesimal generator of (X^π, V) and

$$\Lambda(t, x, v) = M(t, x + (v - e^{(\alpha-r)(t-t_0)}I)^+) = -e^{-\gamma(x+(v-e^{(\alpha-r)(t-t_0)}I)^+}e^{-\frac{(\mu_1-r)^2}{2\sigma^2}(T-t)} \quad (9)$$

is the utility obtained from exercising the investment option at time t. Problem (7) needs to be solved for $(t, x, v) \in [t_0, T) \times \mathbb{R} \times (0, \infty)$, supplemented by the boundary conditions:

$$u(T, x, v) = -e^{-\gamma[x+(v-e^{(\alpha-r)(T-t_0)}I)^+]}$$

$$u(t, x, 0) = -e^{-\gamma x}e^{-\frac{(\mu_1-r)^2}{2\sigma^2}(T-t)}. \quad (10)$$

Because we are using an exponential utility, we see that the term $-e^{-\gamma x}$ can be factored out from expression (5), and consequently from the boundary conditions (9) and (10) as well. Therefore, as suggested in Zariphopoulou (2001), we can write:

$$u(t, x, v) = -e^{-\gamma x}F(t, v)^{\frac{1}{1-\rho^2}}, \quad (11)$$

for a function $F(t, v)$ to be determined. We then find that the corresponding free boundary problem for F becomes:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \mathcal{L}^0 F \geq 0, \\ F(t, v) \leq \kappa(t, v), \\ \left(\dfrac{\partial F}{\partial t} + \mathcal{L}^0 F\right) \cdot (F - \kappa) = 0, \end{cases} \quad (12)$$

where:

$$\mathcal{L}^0 = \left[\mu_2 - r - \rho\frac{\mu_1 - r}{\sigma_1}\sigma_2\right]v\frac{\partial}{\partial v} + \frac{\sigma_2^2 v^2}{2}\frac{\partial^2}{\partial v^2} \quad (13)$$

and

$$\kappa(t, v) = e^{-\gamma(1-\rho^2)(v-e^{(\alpha-r)(t-t_0)}I)^+}. \quad (14)$$

Problem (12) needs to be solved for $(t, v) \in [t_0, T) \times (0, \infty)$, subject to the boundary conditions:

$$F(T, v) = e^{-\gamma(1-\rho^2)(v-e^{(\alpha-r)(T-t_0)}I)^+}$$

$$F(t, 0) = 1. \quad (15)$$

Observe that this free boundary problem is independent of X and S. Accordingly, we define the investor's optimal investment threshold as the function:

$$V^*(t) = \inf\{v \geq 0 : F(t, v) = \kappa(t, v)\} \quad (16)$$

and the optimal exercise time as:

$$\tau^* = \inf\{t_0 \leq t \leq T : V_t = V^\star(t)\}. \quad (17)$$

It follows from the definition (6) and the factorization (11) that the indifference value for the option to invest in the project is given by $p = p(t_0, V_{t_0})$ where:

$$p(t, v) = -\frac{1}{\gamma(1-\rho^2)} \log F(t, v).$$ (18)

Therefore, we can rewrite the original free boundary problem as:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mathcal{L}^0 p - \dfrac{1}{2}\gamma(1-\rho^2)\sigma_2^2 v^2 \left(\dfrac{\partial p}{\partial v}\right)^2 \leq 0, \\[2ex] p(t, v) \geq \left(v - e^{(\alpha-r)(t-t_0)} I\right)^+, \\[2ex] \left[\dfrac{\partial p}{\partial t} + \mathcal{L}^0 p - \dfrac{1}{2}\gamma(1-\rho^2)\sigma_2^2 v^2 \left(\dfrac{\partial p}{\partial v}\right)^2\right] \cdot (p - (v - e^{(\alpha-r)(t-t_0)} I)^+) = 0. \end{cases}$$ (19)

Similarly, we can rewrite the optimal exercise time τ^* in terms of p as follows:

$$\tau^* = \inf\{t_0 \leq t \leq T : p(t, V_t) = (V_t - e^{(\alpha-r)(t-t_0)} I)^+\}.$$ (20)

(iii) Properties of the Optimal Investment Threshold

In this section we investigate how the optimal exercise policy for the option to invest depends on the underlying parameters. We will always assume that the interest rate r, the expected return μ_1 and volatility σ_1 for S_t, the growth rate α for the sunk cost $I(t)$, and the initial sunk cost I are fixed. On the other hand, we treat the risk aversion γ, the correlation ρ, and the underlying project growth rate μ_2 and volatility σ_2 as variable parameters. We then perform comparative statics, that is, we change each of these parameters while keeping the others constant and analyze the corresponding behavior of the optimal investment policy.

Observe that for each choice of values for σ_2 and ρ, the assumption that asset prices are in equilibrium implies a condition on the expected return μ_2 on the project. For example, if we assume (as we do, for simplicity) that S_t is the discounted price of the market portfolio, then the CAPM equilibrium expected rate of return $\bar{\mu}_2$ for a traded asset with volatility σ_2 and correlation ρ should satisfy:

$$\frac{\bar{\mu}_2 - r}{\sigma_2} = \rho\left(\frac{\mu_1 - r}{\sigma_1}\right).$$ (21)

Because the project is not traded, its actual rate of return μ_2 can differ from the equilibrium rate $\bar{\mu}$. The difference $\delta = \bar{\mu}_2 - \mu_2$, known as the *below–equilibrium rate–of–return shortfall* (see McDonald and Siegel, 1984, for a discussion in the context of option pricing), should be interpreted as the incomplete market analogue of a dividend rate paid by the project. For comparison with the complete market case, we take δ to be an underlying parameter, so that μ_2 becomes automatically determined by:

$$\mu_2 = \rho\frac{\mu_1 - r}{\sigma_1}\sigma_2 + r - \delta.$$ (22)

The behavior of the investment threshold with respect to the underlying parameters is established in the next proposition, which we prove using the same technique as in Leung and Sircar (2009), but adapted to the present formulation of the problem.

Proposition 2.1. The optimal exercise boundary shifts:

1. upward as ρ^2 increases;

2. downward as the risk aversion γ increases;

3. downward as the dividend rate δ increases;

Proof: Observe first that it follows from (20) that a smaller indifference value leads to a smaller optimal exercise time, which in turns implies a lower optimal exercise boundary. To establish how the indifference value changes with the underlying parameters, we use the comparison principle for the variational inequality:

$$\min \left\{ -\frac{\partial p}{\partial t} - \mathcal{L}^0 p + \frac{1}{2}\gamma(1 - \rho^2)\sigma_2^2 v^2 \left(\frac{\partial p}{\partial v}\right)^2 , p(t, v) - (v - e^{(\alpha - r)(t - t_0)}I)^+ \right\} = 0, \quad (23)$$

which is known to be equivalent to (19).

1. Recalling the definition of $\mathcal{L}^0$ in (13), we see the variational inequality depends on ρ through the terms:

$$- \left[\mu_2 - r - \rho\frac{\mu_1 - r}{\sigma_1}\sigma_2 \right] v\frac{\partial}{\partial v} + \frac{1}{2}\gamma(1 - \rho^2)\sigma_2^2 v^2 \left(\frac{\partial p}{\partial v}\right)^2 .$$

But using the equilibrium condition (21), we have that:

$$- \left[\mu_2 - r - \rho\frac{\mu_1 - r}{\sigma_1}\sigma_2 \right] v\frac{\partial p}{\partial v} = \delta\frac{\partial p}{\partial v} \qquad (24)$$

so that the dependence on ρ reduces to the nonlinear term:

$$\frac{1}{2}\gamma(1 - \rho^2)\sigma_2^2 v^2 \left(\frac{\partial p}{\partial v}\right)^2 . \qquad (25)$$

Therefore, the indifference value is a symmetric function of ρ, and increases as ρ^2 increases from 0 to 1.

2. Since the nonlinear term (25) is increasing in γ, it follows that p is decreasing in γ.

3. For this item observe first that $\frac{\partial p}{\partial v} \geq 0$, because $u(t, x, v)$ defined in (5) (and consequently $p(t, v)$) is an increasing function v. It then follows that the term (24) is increasing in δ, which implies that p is decreasing in δ. ■

The intuition behind this result is clear. First, as ρ^2 increases, more of the risk associated with the project V_t can be hedged using the financial asset S_t, so that the option to postpone approaches the full value of a call option in a complete market,

leading to a higher exercise boundary. Secondly, as γ increases, the investor becomes more averse towards the remaining idiosyncratic risk in the project, leading to a smaller value for the option to postpone investment and consequently a lower exercise boundary. Finally, analogously to a dividend rate in a complete market, the rate δ measures the opportunity cost of not investing in the project. Therefore, as δ increases, postponing investment becomes more costly, also leading to a lower exercise boundary.

We remark that items 1 and 2 of the previous proposition are the finite–horizon analogues of Proposition 3.5 of Henderson (2007). Finally, observe that the variational inequality (23) depends on σ_2 through the term:

$$-\frac{\sigma_2^2 v^2}{2}\frac{\partial^2 p}{\partial v^2} + \frac{1}{2}\gamma(1-\rho^2)\sigma_2^2 v^2 \left(\frac{\partial p}{\partial v}\right)^2. \tag{26}$$

Since p is a convex function of v (as can be observed, for instance, in Figure 7), the first term above is increasing in σ_2 whereas the second term is decreasing. The resulting contribution is therefore not necessarily monotone in σ_2, since its sign depends in a complicated way on the underlying model parameters and values of v. Therefore, we cannot expect the indifference value to be monotone in σ_2, as demonstrated numerically in the next section.

Proposition 2.2. If $\alpha = r$, then the optimal investment threshold $V^*(t)$ is decreasing in time.

Proof: The solution to problem (12) admits a probabilistic representation (see Oberman and Zariphopoulou, 2003) of the form:

$$F(t, v) = \inf_{\tau \in \mathcal{T}[t,T]} E^0[\kappa(\tau, V_\tau)|V_t = v],$$

where $E^0[\,\cdot\,]$ denotes the expectation operator under the *minimal martingale measure* Q^0 defined by:

$$\frac{dQ^0}{dP} = e^{-\frac{\mu_1-r}{\sigma_1}W_T - \frac{1}{2}\frac{(\mu_1-r)^2}{\sigma_1^2}T}. \tag{27}$$

Setting $\alpha = r$ and using the time–homogeneity of the diffusion V_t, we have that:

$$F(t, v) = \inf_{\tau \in \mathcal{T}[t,T]} E^0\left[e^{-\gamma(1-\rho^2)(V_\tau-I)^+}\big|V_t = v\right]$$

$$= \inf_{\tau \in \mathcal{T}_{t_0, T-t+t_0}} E^0\left[e^{-\gamma(1-\rho^2)(V_\tau-I)^+}\big|V_{t_0} = v\right].$$

For any $s \leq t$ we have that $\mathcal{T}[t_0, T-t+t_0] \subset \mathcal{T}[t_0, T-s+t_0]$, so $F(s, v) \leq F(t, v)$. Now fix $v > 0$ and suppose that it is optimal to exercise at (s, v), that is, $F(s, v) = \kappa(s, v)$. Using the fact that F is increasing in time (as we just established), we have that:

$$\kappa(s, v) = F(s, v) \leq F(t, v) \leq \kappa(t, v). \tag{28}$$

Now since $\alpha = r$ by hypothesis, it follows for the definition of κ in (14) that:

$$\kappa(s, v) = \kappa(t, v) = e^{-\gamma(1-\rho^2)(v-I)^+},$$

leading us to conclude that the string of inequalities (28) are in fact equalities. Therefore, $F(t, v) = \kappa(t, v)$, which means that it is also optimal to exercise at (t, v), in view of (16). But this is equivalent to $V^*(t) \leq V^*(s)$.   ∎

Corollary 2.3. If $\alpha = r$, the optimal investment threshold is an increasing function of the time–to–maturity parameter $(T - t)$. In particular, for a fixed time t_0, we have that the investment threshold $V^*(t_0)$ increases as the maturity T for the option increases.

3. BINOMIAL APPROXIMATION

One approach to compute the indifference value $p(t, V)$ for the option to invest and the corresponding threshold curve $V^*(t)$ is to directly apply a finite–difference approximation to the obstacle problem (19) in the manner described in Oberman and Zariphopoulou (2003). Because of the nonlinear terms appearing in this problem, the usual arguments to prove convergence of the approximation cannot be directly applied. Instead, one can use the concept of viscosity solutions and prove convergence of finite–difference schemes that satisfy an extra condition of *monotonicity*, in addition to the usual *stability* and *consistence* that are sufficient to establish convergence in the linear case.

Alternatively, in view of (18), one can follow Leung and Sircar (2009) and apply a finite–difference approximation to the linear obstacle problem (12). This has the advantage of bypassing the convergence issues associated with the nonlinearity in (19), but at the expense of focusing all the direct computations on the quantity $F(t, y)$, which has no clear financial interpretation.

In the spirit of Cox et al. (1979), we prefer to work with a simplified binomial model approximation instead. The convergence of such approximations for risk–neutral prices of European contingent claims was established for a large class of diffusion process in Nelson and Ramaswamy (1990), whereas the corresponding result for American claims was established in Amin and Khanna (1994). These results, however, cannot be directly applied to the indifference value $p(t, V)$ because of the nonlinearity involved in (19). As a consequence, discrete–time indifference values arising in binomial models have been studied in their own right, for instance in Musiela and Zariphopoulou (2004), without explicit consideration of their continuous time limit.

Our approach in what follows will be to directly formulate the real option problem in incomplete markets in a discrete–time framework, where an explicit valuation algorithm can be obtained. Next we explain how to adjust the model parameters so that in the limit the underlying stochastic processes used in the binomial model converge to the corresponding continuous–time model specified by (1). We then compute indifference values and investment thresholds using these parameters and verify that they exhibit the properties established in Propositions 2.1 and 2.2, while deferring the delicate question of rigorously establishing convergence of the approximation to future work.

(i) Investment Decisions in One Period

Consider an investor who needs to decide whether to pay a sunk cost $I_0 = I$ for a project with current value V_0. Assume that such investment can be made either at time 0 or postponed until time T, when the sunk cost will be $I_T = e^{\alpha T} I$ and the project

value might rise or fall according to specified probabilities. The opportunity to invest is then formally equivalent to a discrete-time American call option with strike price I_i, $i = 0$, T, and having the project value as the underlying asset.

As before, let us assume the existence of a riskless cash account with constant annualized interest rate r, which we use as a fixed *numeraire*. Denote the discounted project value by V and the discounted price of a correlated traded financial asset by S. We then specify their one–period dynamics by:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{29}$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.

Without the opportunity to invest in the project V, a rational investor with initial wealth x will keep an amount ξ in the cash account and hold H units of the traded asset S in such a way as to maximize the expected utility of discounted terminal wealth:

$$X_T^{x,H} = \xi + HS_T = x + H(S_T - S_0). \tag{30}$$

That is, the investor will try to solve the optimization problem:

$$M(x) = \max_{H \in \mathbb{R}} E\big[U\big(X_T^{x,H}\big)\big], \tag{31}$$

which is the one–period analogue of (4).

Suppose next that the investor pays an amount p for the opportunity to invest in the project at the end of the period, thereby receiving a discounted payoff $C_T = (V_T - e^{(\alpha - r)T}I)^+$. In other words, an investor with initial wealth x who acquires the option for the price p will try to solve the modified optimization problem:

$$u(x - p) = \sup_{H \in \mathbb{R}} E\big[U\big(X_T^{x-p,H} + C_T\big)\big]. \tag{32}$$

As before, we define the indifference value for the option to invest in the final period as the amount p^c that solves the equation:

$$M(x) = u(x - p^c). \tag{33}$$

Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, it is then a straightforward calculation to show that, for an exponential utility, such indifference value is given by:

$$p^c = g(C_h, C_\ell) \tag{34}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by:

$$g(x_1, x_2) = \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right), \quad (35)$$

with:

$$q = \frac{1-d}{u-d}.$$

We will henceforth refer to p^c as the *continuation value* for holding the option of investing at a later time. If we now introduce the possibility of investment at time $t = 0$, it is clear that immediate exercise of this option will occur whenever its *exercise value* $(V_0 - I)^+$ is larger than its continuation value p^c. That is, from the point of view of this investor, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by:

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{(\alpha - r)T}I)^+, (\ell V_0 - e^{(\alpha - r)T}I)^+)\}. \quad (36)$$

(ii) The Multiperiod Model

An approximation for the continuous–time market (1) can be obtained by dividing the time interval $[0, T]$ into N subintervals with equal time steps $\Delta t = T/N$ and taking the one–period dynamics for the discrete–time processes (S_n, V_n) to be given by (29). We then need to choose the dynamic parameters u, d, h, ℓ and the one-period probabilities p_i so that, in the limit of small Δt, such dynamics matches the distributional properties of the continuous time processes S_t and V_t.

To avoid unnecessary complications due to nonlinearities in the calibration of the Geometric Brownian motions in (1), we follow Brandimarte (2006) and work with their logarithms $Y_t^1 = \log S_t$ and $Y_t^2 = \log V_t$ instead. It then follows that:

$$dY_t^1 = v_1 dt + \sigma_1 dW_t^1$$
$$dY_t^2 = v_2 dt + \sigma_2\left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right), \quad (37)$$

where $v_i = \mu_i - r - \sigma_i^2/2$. Assuming for simplicity that $u = 1/d$ and $h = 1/\ell$ and denoting the logarithmic increments by $\Delta y_1 = \log u$ and $\Delta y_2 = \log h$, all we need in order to guarantee weak convergence of (Y_n^1, Y_n^2) to (Y_t^1, Y_t^2) is to find parameters such that the mean and covariance matrix for the discrete-time process on the two–dimensional binomial tree with increments $(\Delta y_1, \Delta y_2)$ match those of the continuous–time process in (37) up to order Δt. In other words, we need to verify that:

$$E[\Delta Y^1] := [p_1 + p_2 - p_3 - p_4]\Delta y_1 = v_1 \Delta t \quad (38)$$

$$E[\Delta Y^2] := [p_1 - p_2 + p_3 - p_4)]\Delta y_2 = v_2 \Delta t \quad (39)$$

$$E[(\Delta Y^1)^2] := [p_1 + p_2 + p_3 + p_4](\Delta y_1)^2 = \sigma_1^2 \Delta t \quad (40)$$

$$E[(\Delta Y^2)^2] := [p_1 + p_2 + p_3 + p_4](\Delta y_2)^2 = \sigma_2^2 \Delta t \quad (41)$$

$$E[\Delta Y^1 \Delta Y^2] := [p_1 - p_2 - p_3 + p_4]\Delta y_1 \Delta y_2 = \rho \sigma_1 \sigma_2 \Delta t. \tag{42}$$

Supplemented by the condition that probabilities add up to one, we are left with six equations to be solved for the six unknowns (p_1, p_2, p_3, p_4) and $(\Delta y_1, \Delta y_2)$. Fortunately the nonlinear equations (40) and (41) readily yield:

$$\Delta y_1 = \sigma_1 \sqrt{\Delta t} \tag{43}$$

$$\Delta y_2 = \sigma_2 \sqrt{\Delta t} \tag{44}$$

and we are left with a linear system of the form $\mathbf{Ap} = \mathbf{b}$ where:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \dfrac{v_1 \sqrt{\Delta t}}{\sigma_1} \\ \dfrac{v_2 \sqrt{\Delta t}}{\sigma_2} \\ \rho \\ 1 \end{bmatrix}. \tag{45}$$

It is then easy to verify that the unique solution for the linear system is:

$$p_1 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(\frac{v_1}{\sigma_1} + \frac{v_2}{\sigma_2}\right)\right] \tag{46}$$

$$p_2 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(\frac{v_1}{\sigma_1} - \frac{v_2}{\sigma_2}\right)\right] \tag{47}$$

$$p_3 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(-\frac{v_1}{\sigma_1} + \frac{v_2}{\sigma_2}\right)\right] \tag{48}$$

$$p_4 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(-\frac{v_1}{\sigma_1} - \frac{v_2}{\sigma_2}\right)\right]. \tag{49}$$

Since the weak convergence above also holds for $(S_t, V_t) = (e^{Y_t^1}, e^{Y_t^2})$, we can work with the actual prices instead of their logarithm in the tree itself, that is, we work with the increments:

$$u = e^{\Delta y_1} = e^{\sigma_1 \sqrt{\Delta t}}, \qquad d = 1/u = e^{-\sigma_1 \sqrt{\Delta t}} \tag{50}$$

$$h = e^{\Delta y_2} = e^{\sigma_2 \sqrt{\Delta t}}, \qquad \ell = 1/h = e^{-\sigma_2 \sqrt{\Delta t}}. \tag{51}$$

Moreover, since both the continuation and the exercise values in (36) depend only on V, we do not actually need to keep track of a fully fledged two–dimensional binomial tree. That is, from now on we can proceed as if we only had to compute prices on a binomial tree for the asset V. This is not to say that the traded asset S plays no role in the valuation of the option to invest in the project V: the expected return and

volatility of S, along with its correlation with V, are used to calculate the probabilities in (46)–(49), which are in turn necessary to compute the continuation value (34).

Having fixed these parameters, let us choose a sufficiently large integer M and denote:

$$V^{(i)} = h^{M+1-i} V_0, \qquad i = 1, \ldots, 2M+1. \tag{52}$$

These values range from $(h^M V_0)$ to $(\ell^M V_0)$, respectively the highest and lowest achievable discounted project values starting from the middle point V_0 with the multiplicative parameter $h = \ell^{-1} > 1$. In practice, M should be chosen so that the highest and lowest values are comfortably beyond the range of project values that can be reached during the time interval $[0, T]$ with reasonable probabilities (for instance, returns which are away from their mean by more than three or four standard deviations). Each realization for the discrete-time process V_n following the dynamics (29) can then be thought of as a path over a $(2M+1) \times N$ rectangular grid having the values (52) as its repeated columns.

The discounted value of the option to invest on the project can then be determined as a function $C_{i,n}$ on this grid, with the index $i = 1, \ldots, 2N+1$ referring to the underlying project value $V^{(i)}$, and the index $n = 0, \ldots, N$ referring to time $t_n = n\Delta t$. We start by specifying the following boundary conditions:

$$C_{i,N} = (V^{(i)} - e^{(\alpha-r)T} I)^+, \qquad i = 1, \ldots, 2N+1, \tag{53}$$

$$C_{1,n} = V^{(1)} - e^{(\alpha-r)n\Delta t} I, \qquad n = 0, \ldots, N, \tag{54}$$

$$C_{2N+1,n} = 0, \qquad n = 0, \ldots, N. \tag{55}$$

The terminal condition (53) corresponds to the fact that at maturity the option to invest should be exercised whenever the project value exceeds the investment cost. The top boundary condition (54) means that such option should always be exercised when the project value is at its highest. The bottom boundary condition (55) corresponds to the fact that the option is worthless when the project is at its lowest. The values in the interior of the grid are then obtained by backward induction as follows:

$$C_{i,n} = \max \left\{ (V^{(i)} - e^{(\alpha-r)n\Delta t} I)^+, g(C_{i+1,n+1}, C_{i-1,n+1}) \right\}, \quad \begin{array}{l} n = N-1, \ldots, 0 \\ i = 2, \ldots, 2N. \end{array} \tag{56}$$

That is, at each node on the grid, the investor chooses between exercising the investment option, obtaining its immediate exercise value $V^{(i)} - e^{(\alpha-r)n\Delta t} I$, or holding the option one step into the future, retaining its continuation value $g(C_{i+1,n+1}, C_{i-1,n+1})$.

Accordingly, at each time t_n, the exercise threshold V_n^* is defined as the project value for which the exercise value for the option becomes higher than its continuation value. For project values below V_n^*, the investor will prefer to hold the option, while for project values higher than such threshold, preference for immediate exercise will prevail.

4. NUMERICAL EXPERIMENTS

We now confirm the theoretical results of Section 2*(iii)* by implementing the algorithm described in the previous section and investigating how the exercise threshold, and consequently the value of the option to invest, varies with different model parameters. Specifically, we compute the recursive formula (56) supplemented by the boundary conditions (53)–(55) using the parameters specified by (46)–(49) and (50) with a vector of project values (52). In all of the following numerical experiments, we used a fixed time step $\Delta t = 1/2500$, so that the relative precision for project values on the grid is of the order $\sigma_2 \sqrt{\Delta t} \sim 0.004$. For each point marked in the pictures below, the thresholds and option values were obtained on a typical 500×25000 grid in approximately 10 seconds using a desktop computer at 3GHz.

In what follows, unless explicitly indicated, we use the following fixed parameters:

$$I = 1, \qquad r = 0.04, \quad \delta = 0.04, \quad \alpha = 0, \quad T = 10$$

$$\mu_1 = 0.115, \quad \sigma_1 = 0.25, \quad \sigma_2 = 0.2. \tag{57}$$

As we mentioned before, fixing these parameters has the effect of automatically determining the return rate μ_2 for each choice of correlation ρ according to the formula (22).

(i) Correlation

Because incompleteness is the main theme of this paper, we start with the dependence on correlation. In accordance with item 1 of Proposition 2.1, Figure 1 shows that the exercise threshold increases symmetrically as the correlation moves away from $\rho = 0$ towards $\rho = \pm 1$, meaning that the possibility to hedge the risk in the real asset using the correlated traded asset increases the value of the option to invest.

Observe further that the limits $\rho \to \pm 1$ in our model correspond to a complete market, since options on the underlying asset V can then be perfectly replicated by trading in the asset S. In this case, the infinite-horizon investment threshold can be determined through risk–neutral arguments (see proposition 4.1 in Henderson, 2007) and is given by the Dixit–Pindyck formula:

$$V_{DP}^* = \frac{\beta_{DP}}{(\beta_{DP} - 1)} I, \tag{58}$$

where β_{DP} is the positive solution to the quadratic equation:

$$\frac{1}{2} \sigma_2^2 \beta (\beta - 1) + (\mu_2 - \lambda \sigma_2) \beta - r = 0. \tag{59}$$

Using the parameters (57) and setting $\rho = 1$ in (22) (the case $\rho = -1$ is treated similarly by taking the opposite position in the traded asset), the positive root for this quadratic equation and the corresponding exercise threshold are given by:

Figure 1

Exercise Threshold as a Function of Correlation for Different Levels of Risk Aversion
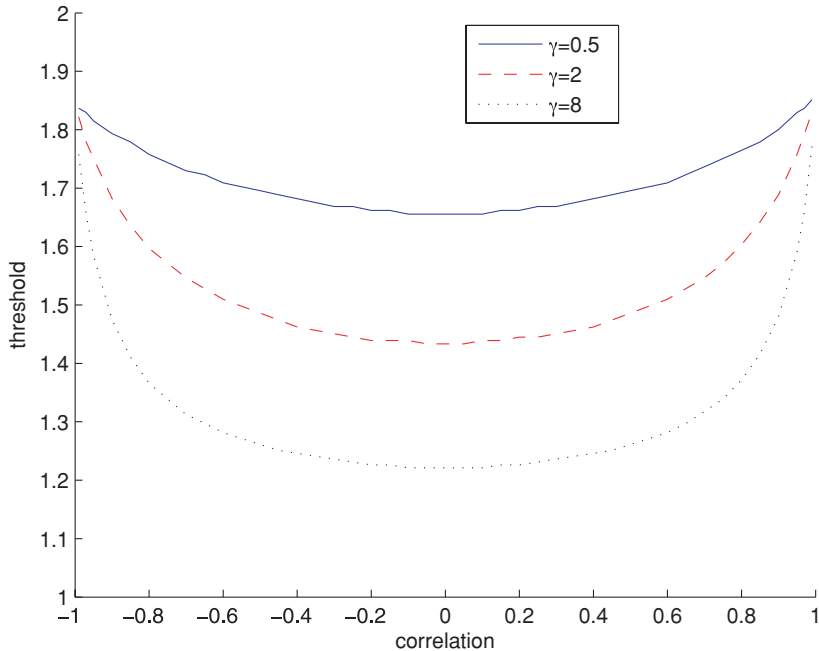
Note:
Model Parameters are: $I = 1$, $r = 0.04$, $\delta = 0.04$, $\alpha = 0$, $T = 10$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$.

$$\beta_{DP} = \frac{1}{2} - \frac{r - \delta}{\sigma_2^2} + \sqrt{\left(\frac{r - \delta}{\sigma_2^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma_2^2}} = 2.$$

$$V_{DP}^* = 2 \tag{60}$$

By contrast, the investment threshold obtained from a simple net present value criterion (that is, invest whenever the NPV for the project is positive) in this case is equal to $V_{NPV}^* = 1$. This constitutes the most widespread result from real option theory: irreversibility and time flexibility lead to investors waiting until the project value reaches much larger thresholds before committing to an investment decision. Interestingly, we see from Figure 1 that this remains largely true in incomplete markets, since even at its minimum, corresponding to $\rho = 0$, the investment threshold is still higher than what is suggested by NPV. That is, even when the risk in the project is entirely idiosyncratic and therefore cannot be hedged with financial assets, time flexibility still confers an option value to the opportunity to invest which is higher than its net present value, *irrespectively of any replication argument*. As $\rho \to \pm 1$, we see that the exercise threshold converges to a constant that does not depend on the risk aversion, since the investment decision in a complete market follows the risk–neutral valuation mentioned above. The limits observed in the figure are strictly smaller than the infinite–horizon threshold $V_{DP}^* = 2$ since we are using the finite time $T = 10$ in this example.

(ii) Risk Aversion

We can already see indirectly in Figure 1 that higher risk aversion leads to lower investment thresholds, confirming the result in item 2 of Proposition 2.1. This effect is confirmed more explicitly in Figure 2 , where we clearly see the investment threshold as a decreasing function of the parameter γ. That is, risk aversion can significantly erode the option value obtained from time flexibility. In the limit $\gamma \to \infty$, one can explicitly show that expression (35) tends to the subhedge price of the derivative, which is zero for a call option, so that the value for the investment opportunity reduces to its net present value, and the investment threshold collapses to $V_{NPV}^* = 1$. As we observe in the graph, this erosion of value with risk aversion is faster for lower correlations between the project and the traded asset.

As observed in Henderson (2007), the limit $\gamma \to 1$ corresponds to the exercise threshold obtained in McDonald and Siegel (1986) in an equilibrium context and infinite–horizon setup by assuming that investors require compensation for market risks whilst being risk–neutral towards idiosyncratic risk. With the equilibrium rate on the project given by (21) when the traded asset is the market portfolio, the McDonald and Siegel threshold is given by (see Proposition 4.2 in Henderson, 2007):

$$V_{MS}^* = \frac{\beta_{MS}}{(\beta_{MS} - 1)} I, \tag{61}$$

where β_{MS} is the positive solution to the quadratic equation:

Figure 2

Exercise Threshold as a Function of Risk Aversion for Different Levels of Correlation
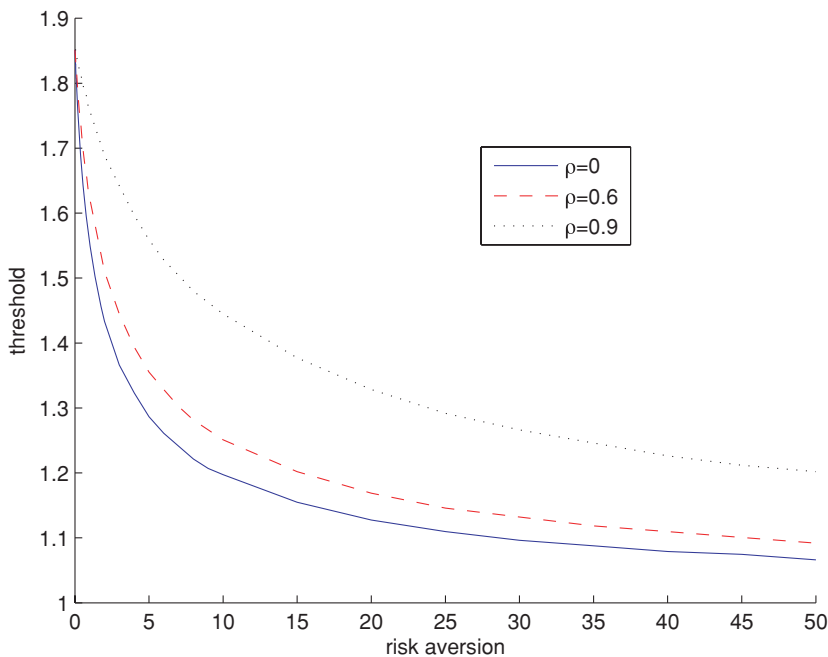
Note:
Model Parameters are: $I = 1, r = 0.04, \delta = 0.04, \alpha = 0, T = 10, \mu_1 = 0.115, \sigma_1 = 0.25, \sigma_2 = 0.2$.

$$\frac{1}{2}\sigma_2^2\beta(\beta-1)+(\mu_2-\lambda\rho\sigma_2)\beta-r=0. \tag{62}$$

Using the parameters (57) and taking (22) into account, the positive solution to this quadratic equation and the corresponding investment threshold are given by:

$$\beta_{MS}=\frac{1}{2}-\frac{r-\delta}{\sigma_2^2}+\sqrt{\left(\frac{r-\delta}{\sigma_2^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma_2^2}}=2.$$
$$V_{MS}^*=2 \tag{63}$$

We can observe this limiting behavior in Figure 2. In particular, notice that the limiting threshold for $\gamma \to 0$ is independent of the correlation because of our choice of μ_2 satisfying (22). Also notice that our limit is smaller than the infinite–horizon asymptotic value $V_{MS}^*=2$, since $T=10$ for this example.

(iii) Dividend Rate

As explained before, we treat the quantity $\delta = \bar{\mu}_2 - \mu_2$ as the incomplete market analogue of a dividend rate paid by the project, since it measures the difference between the equilibrium return rate $\bar{\mu}_2$ predicted by CAPM according to (21) and the actual return rate obtained by investing in the project. In this way, δ measures the attractiveness of actually investing in the project, as opposed to holding the option to invest. As δ increases, Figure 3 shows that our threshold decreases monotonically as predicted by item 3 of Proposition 2.1.

In the complete market case, it is well–known that if $\delta = 0$ then investment in the project will never occur before T, since in this case the option to invest is equivalent to an American call option on a non–dividend–paying asset, which should never be exercised before maturity. For an infinite time horizon, this is reflected in the fact that as $\delta \to 0$ in (58), we have that $\beta_{DP} \to 1$ and consequently $V_{DP}^* \to \infty$, implying that the option to invest will never be exercised. Setting $\mu_2 = r + \lambda\sigma_2 - \delta$ (that is, using (22) with $\rho = 1$), we see that the condition $\delta > 0$ is equivalent to $\xi < \lambda$, where ξ and λ are the corresponding Sharpe ratios for the project and the market portfolio, given by:

$$\xi=\frac{\mu_2-r}{\sigma_2}, \quad \lambda=\frac{\mu_1-r}{\sigma_1}. \tag{64}$$

A similar analysis holds for the McDonald and Siegel (1986) model. Namely, for $\gamma \to 0$, investment can occur only if $\xi < \lambda\rho$. Using (22), we see that this is equivalent to $\delta > 0$. The surprising feature in Henderson (2007) is that, for an incomplete market and nonzero risk aversion, investment in the project can occur provided $\xi < \lambda\rho + \sigma_2/2$ (see Figure 1 in Herderson, 2007). Using (22), we see that this is equivalent to $\delta > -\frac{\sigma_2^2}{2}$. In other words, for projects with Sharpe ratios $\xi \in [\lambda\rho, \lambda\rho + \sigma_2/2)$, or equivalently for $\delta \in (-\frac{\sigma_2^2}{2}, 0]$, the incomplete market model with nonzero risk aversion described in Henderson (2007) predicts investment in the project once its value reaches a finite threshold V_H^*, whereas the model of McDonald and Siegel (1986) predicts that investment should be postponed indefinitely (or equivalently, that the investment threshold is infinite). As explained in Henderson (2007), in an incomplete market, risk aversion will propel the investor to exercise the investment option even in

Figure 3

Exercise Threshold as a Function of the Dividend Rate for Different Levels of
Correlation
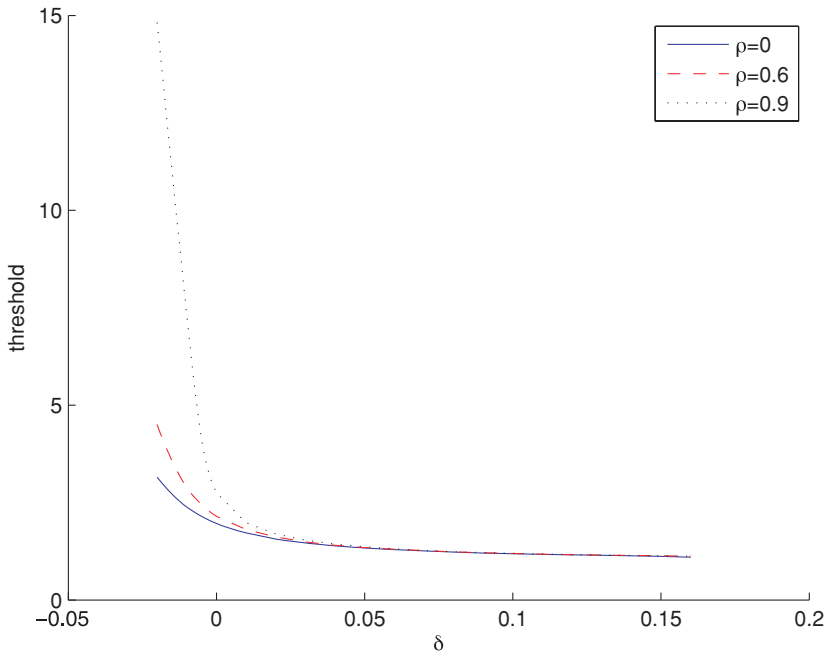
Note:
Model Parameters are: $I = 1$, $r = 0.04$, $\alpha = 0$, $T = 10$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$, $\gamma = 0.5$.

face of a slightly unfavorable Sharpe ratio for the project, or equivalently a moderately negative δ.

We can confirm this behavior also in the finite–horizon case. For $\sigma_2 = 0.2$, $r = 0$, $\gamma = 0.5$ and $T = \infty$, the model in Henderson (2007) predicts a finite investment threshold for $\delta > -0.02$. In Figure 3 we plot our investment threshold as a function of δ for $\sigma_2 = 0.2$, $r = 0$, $\gamma = 0.5$, $T = 10$ and the remaining parameters as in (57). As expected, the threshold remains finite for moderately negative δ, but increases rapidly as $\delta \to -0.02$.

(iv) Volatility

Another unexpected feature observed in Henderson (2007) is that the investment threshold and the corresponding value for the option to invest are not necessarily increasing functions of the project volatility. As we mentioned after Proposition 2.1, this behavior can also happen in the finite–horizon case, since the variational inequality (23) is not necessarily monotone in σ_2. This is surprising, since classical real options analysis predicts that high uncertainty in the project results in a high value for the option to postpone, leading to a high investment threshold and consequent delayed investments.

As can be inferred from Proposition 5.1 of Henderson (2007), the key determinant for the behavior of the threshold with respect to the underlying project volatility is

Figure 4

Exercise Threshold as a Function of the Project Volatility for Different Dividend Rates
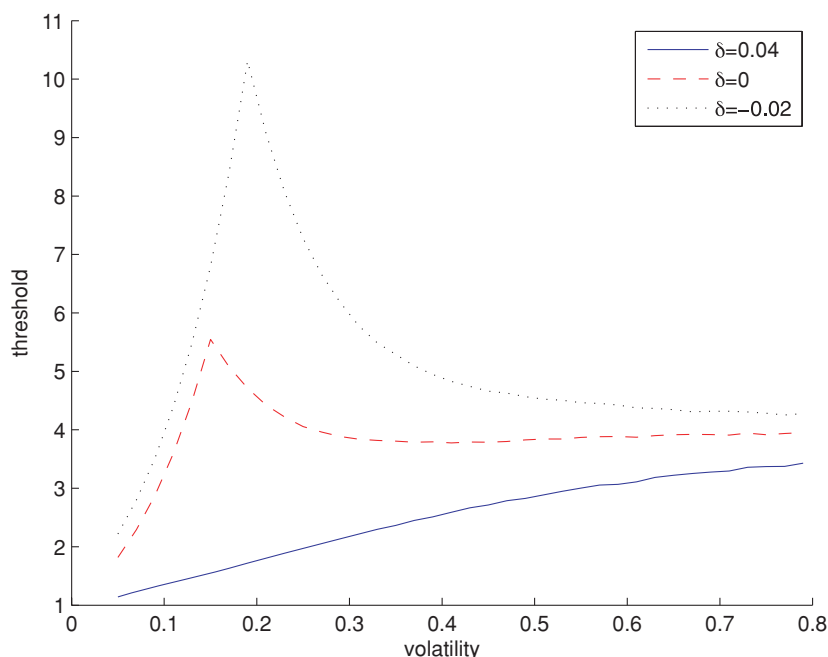
Note:
Model Parameters are: $I = 1$, $r = 0.04$, $\alpha = 0$, $T = 10$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\rho = 0.9$, $\gamma = 1$.

the difference between the Sharpe ratios for the project and the market portfolio, which is equivalently expressed by the below–equilibrium shortfall rate δ. For large enough values of δ the convexity of the option pay-off is the dominant effect and the threshold increases with volatility due to Jensen's inequality, similarly to what happens in a complete market. For smaller values of δ the concavity of the utility function is the dominant effect and the threshold initially increases with volatility and then decreases sharply because of aversion to the risk associated with high volatility. This is clearly demonstrated in Figure 4 , obtained with $\rho = 0.9$, $\gamma = 1$, the three indicated values for δ, and the remaining parameters as in (57).

(v) Time to Maturity

To investigate the dependence with the maturity T, we first calculate the infinite–maturity thresholds according to equation (6) in Henderson (2007) and compare it with the finite–maturity thresholds obtained in our model with $\alpha = r = 0$. The results are shown in Figures 5 and 6 for different levels of correlation and risk aversion, confirming the result of Corollary 2.3. We also see that the exercise threshold can take a long time to converge to its asymptotic value, shown as the horizontal lines, particularly in the desirable cases of low risk–aversion and high–correlation. This indicates that for typical maturities of only a couple of years the stationary solution provides a poor approximation for its finite–horizon counterpart.

Figure 5

Exercise Threshold as a Function of Time to Maturity for Different Levels of
Correlation
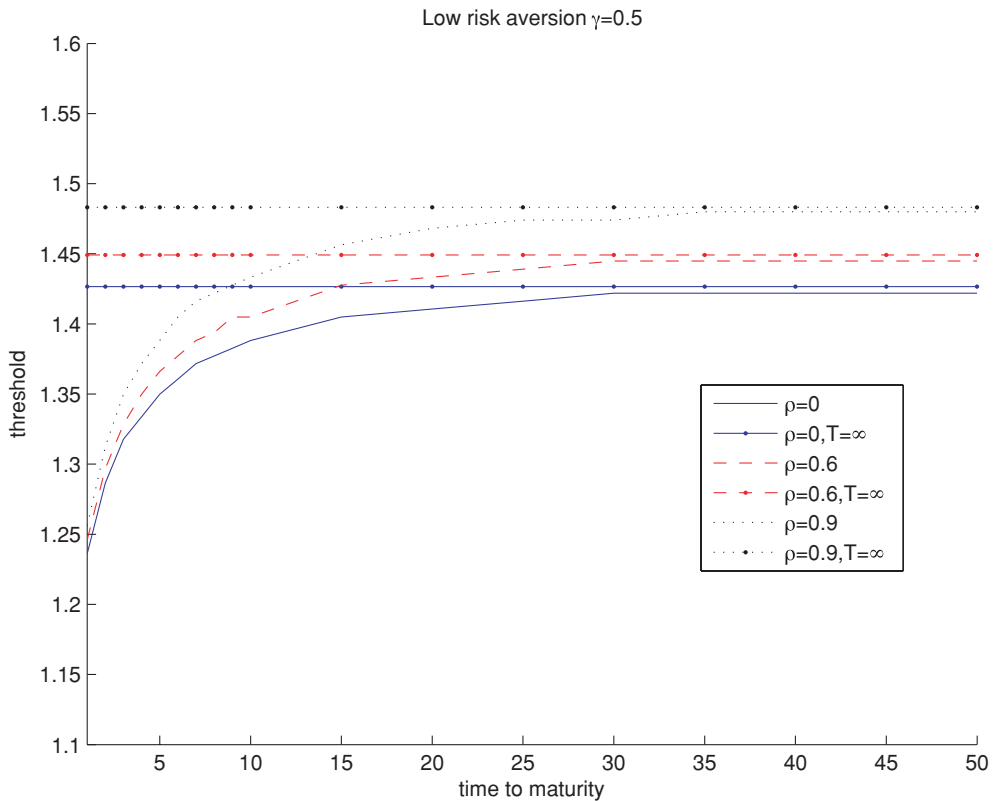
Note:
Model Parameters are: $I = 1$, $r = 0.04$, $\delta = 0.04$, $\alpha = 0$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$, $\gamma = 0.5$.
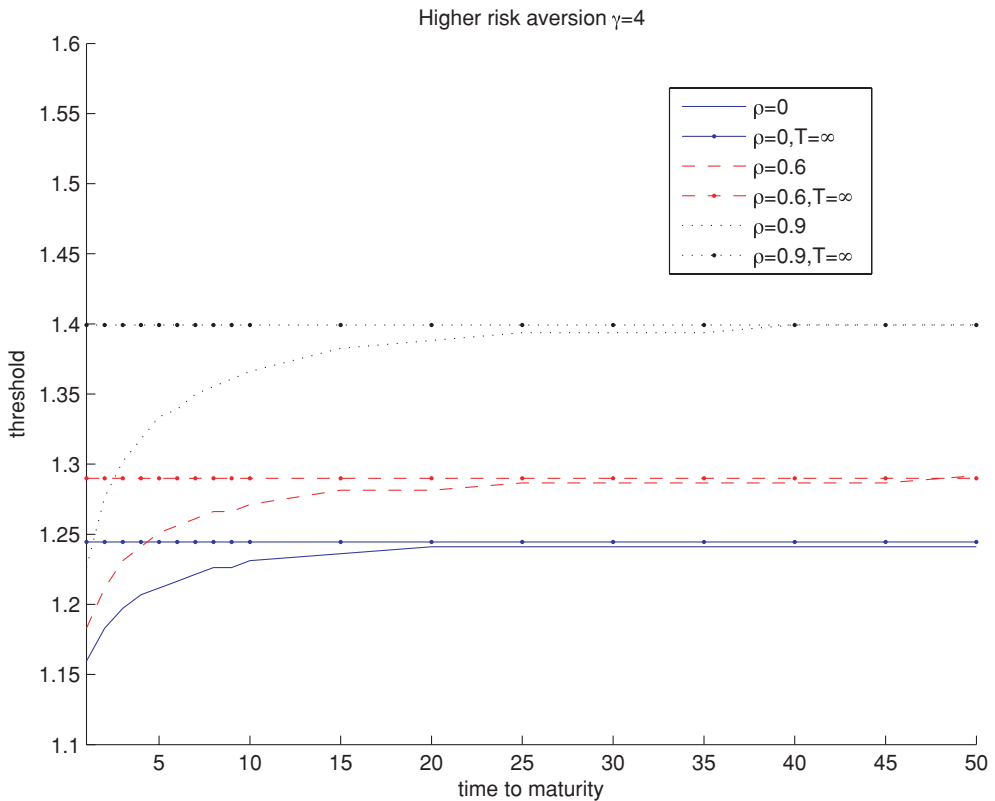
(vi) Option Value

We conclude this section with a graph of the option value as a function of the current
level of the underlying project, presented in Figure 7 . In this graph we use $\gamma = 10$ and
the other fixed parameters as in (57), except for the interest rate, which we set to $r =
0$ in order to compare with Figure 2 of Henderson (2007).

The thresholds obtained in Henderson (2007) for $\rho = 0$ and $\rho = 0.99$ are
respectively 1.1581 and 1.4665, whereas ours are 1.1503 and 1.4238. The difference
is a result of our finite time–to–maturity $T = 10$, in accordance with Corollary 2.3.

For further comparison, we have that Figure 7 is also the qualitative analogue of
Figure 5.3 on page 154 of Dixit and Pindyck (1994). The complete market threshold
calculated according to (58) with $r = 0$ (as opposed to $r = 0.04$ in Dixit and Pindyck,
1994) is $V_{DP}^* = 1.5$. We observe that even for a high correlation $\rho = 0.99$, the incomplete
market thresholds are noticeably smaller than the complete market one, both for $T =
\infty$ and $T = 10$. The difference in this case is a result of risk aversion, in accordance
with item 2 of Proposition 2.1

Figure 6

Exercise Threshold as a Function of Time to Maturity for Different Levels of
Correlation

Higher risk aversion $\gamma=4$

Note:
Model Parameters are: $I = 1$, $r = 0.04$, $\delta = 0.04$, $\alpha = 0$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$, $\gamma = 4$.

As for the option values themselves, we can see that they are higher for higher
correlations. Moreover, we confirm our previous observation that even for $\rho = 0$ the
opportunity to invest is more valuable than its net present value, represented in the
graph by the solid line depicting the function $(V - I)^+$. Finally, notice how the smooth
pasting and matching conditions, which were not *a priori* assumed in our model, are
satisfied by the option values, in the sense that the curves in Figure 7 smoothly match
the function $(V - I)^+$ at the corresponding exercise thresholds, marked in the graph
by the two vertical dotted lines.

5. DISCUSSION

We have proposed a continuous–time model for assessing the value of a finite–maturity
option to invest on a project in the absence of a perfectly spanning financial asset.
We then rigorously established that the exercise thresholds obtained from our model
exhibit the expected qualitative dependence with respect to correlation, uncertainty,

Figure 7

Option Value as a Function of Underlying Project Value
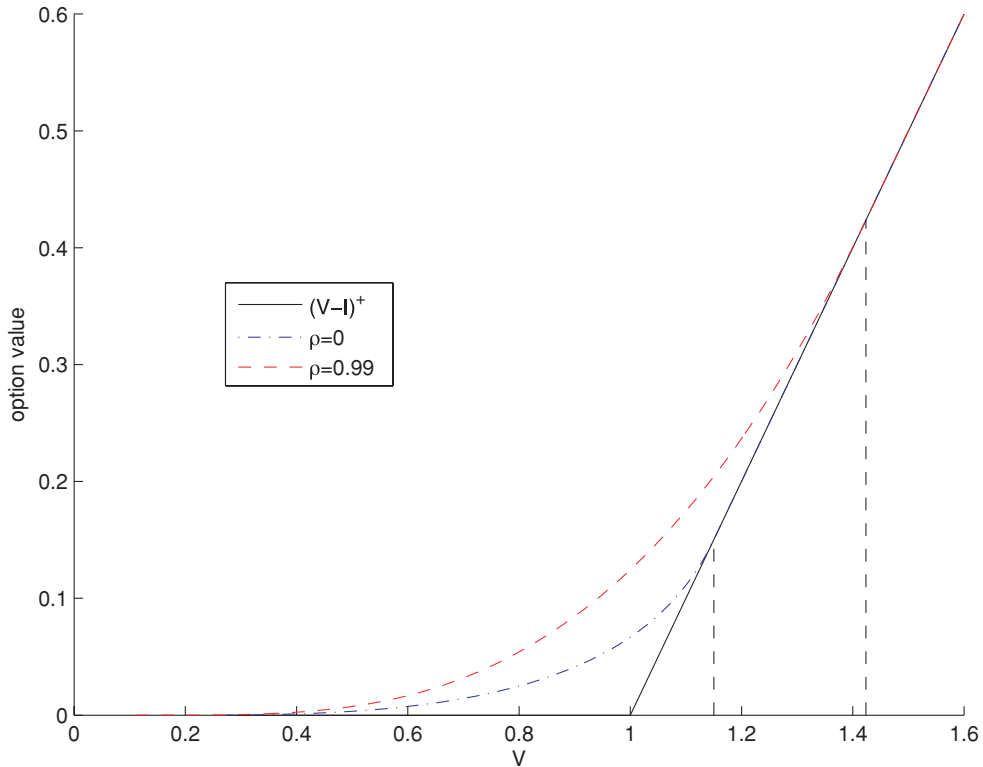
Notes:
The Threshold for $\rho = 0$ and $\rho = 0.99$ are 1.1503 and 1.4238, respectively (shown as vertical dotted lines in the graph).
Model Parameters are: $I = 1$, $r = 0$, $\delta = 0.04$, $\alpha = 0$, $T = 10$, $\mu_1 = 0.115$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$, $\gamma = 10$.

risk aversion, dividend rates and time to maturity. Because of the lack of analytic expressions, we use a multiperiod binomial approximation to verify these properties numerically.

In particular, we verify that even in the zero correlation case, whereby none of the risk in the project can be hedged in a financial market, the paradigm of real options can still be applied to value an investment decision, since the opportunity to invest still carries an option value above its net present value. In other words, it is time flexibility itself, more than the possibility of replication, that is the source of the extra value of an investment opportunity. This value, however, quickly erodes at higher levels of risk aversion, and even more so when the project is uncorrelated to financial markets.

We now compare our results with the related literature. Apart from the outright use of risk–neutral valuation even when markets are incomplete – under the wishful assumption that they are complete enough for all practical purposes – the most widespread alternative method for dealing with incompleteness in a real options context is through the use of dynamic programming with an exogenous discount rate.

This is the approach indicated, for instance, in the second half of Chapter 5 in Dixit and Pindyck (1994), in which an investor equates the expected capital appreciation from a project to the expected rate of return on the investment opportunity, using a corporate rate of return, which is different from the risk–free interest rate and meant to express corporate risk preferences. Despite its popularity, such approach has the serious theoretical drawback that the fully nonlinear risk preferences of a corporation can hardly be expressed through a single discount factor. In fact, instead of being a substitute for a utility function, an exogenous discount factor can be used *together* with a utility function to model risk preferences through time (see for example, Hugonnier and Morellec, 2007), as an alternative to working in terms of discounted wealths and cash–flows as we have done in this paper. At a more practical level, this dynamic programming approach with a corporate discount rate obscures the most important aspects of real options, namely the intuition that can be gained when managerial decisions are treated as options. For example, under the option paradigm, investment on a multi-stage project is analogous to a portfolio of options, each having its own value and interacting in a complex manner towards the value of the whole project. Precisely because such analogies are completely lost in the dynamic programming approach, authors such as Dixit and Pindyck dropped it in the remainder of their book in favor of a contingent claim analysis, which then formally relies back on the complete market framework with a spanning asset hypothesis.

By comparison, our proposed method handles incompleteness by explicitly introducing risk preferences in an economically sound utility–based framework for the realistic case of a finite time horizon, while retaining the computational complexity of a standard binomial valuation. The use of risk preferences in the context of investment decisions appeared, for example, in earlier works of Constantinides (1978) and Smith and Nau (1995), but was restricted to the case of a European–style decision to be made at a fixed expiration time. Our method, on the other hand, addresses the problem of investment decisions that can be made at any intermediate time by modeling them as American contingent claims in incomplete markets. In this respect, it presents an alternative to the numerical methods proposed in Oberman and Zariphopoulou (2003), where the indifference value of an early–exercise claim is characterized as the viscosity solution to a nonlinear variational inequality, and the example of an American put option is computed though using a finite–difference method. Apart from being simpler than the general numerical schemes proposed in Oberman and Zariphopoulou (2003), our method can be easily extended to the case of several interconnected options, therefore providing incomplete market versions for all the standard managerial decisions treated as real options, such as the combined investment and disinvestment options discussed in Stark (2000). For example, the thresholds for investment, abandonment, suspension and reactivation of a project in an incomplete market can all be obtained by a simple extension of the algorithm presented here. All that is necessary is to calculate the project value in each of its active, idle, or suspended phases according to (56), taking into account that the exercise values in a given phase are the continuation values on the other two phases minus the sunk cost for switching phases.

We conclude with a word about implementation. We implicitly assume that the parameters μ_1, σ_1 and r can be obtained from standard estimation techniques applied to the available time series for the market portfolio S_t and some proxy for the risk–free interest rate. Estimating the parameters μ_2, σ_2 and ρ might not be so straightforward

and could require a combination of available time series for the project value V_t and subjective forecasts of its growth rate, underlying volatility and correlation with the market portfolio. The remaining task is then to choose the risk aversion parameter γ, which should reflect the risk preferences for the company. One starting point for this could be the implied risk aversion prevailing in the market, which can be estimated in a variety of ways. In complete markets, they can be easily estimated from option data, using the fact that the pricing kernel (or state price density) encodes information about the utility function of a representative agent (see for example, Jackwerth, 2000). Such estimates, while providing a first approximation for the risk aversion, might not be adequate to the needs of a particular company, since they reflect average market views, rather than the company's risk attitudes. Alternatively, decision makers within a particular company could engage in a self–assessment exercise in order to determine an appropriate risk aversion parameter. In this respect, there is a large literature on how to determine risk aversion from the results of surveys involving specified lotteries (see for example, Kagel and Roth, 1995). Ultimately, several different estimates for each input parameter should be used in the valuation algorithm before a specific investment decision is made. Armed with the sensitivity analysis provided by the results in Section 2*(iii)* and the type of graphs presented in Section 4, a manager can then make well–informed decisions within several alternative scenarios.

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