A stock-flow consistent macroeconomic model with heterogeneous agents: the master equation approach

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May 07, 2018

Abstract

We propose a mean-field approximation to a stock-flow consistent agent-based macroeconomic model with heterogeneous firms and households. Depending on their investment elasticity to past profits, firms can be either aggressive or conservative. Conversely, households are divided into investor and non-investor groups, depending on whether or not they invest a portion of their wealth in the stock market. Both firms and households dynamically change their type according to transition probabilities specified exogenously. The mean-field approximation consists of homogenizing the balance-sheet variables for agents (firms or households) of the same type and compute the time evolution of the corresponding average as a combination of the deterministic dynamic, derived from investment and consumption decisions before a change of type, and the probabilistic change in type, with an appropriate rebalancing to take stock-flow consistency into account. The last step of the approximation consists in replacing the underlying Markov chain with a continuous-time diffusive limit. We present numerical experiments showing the accuracy of the approximation and the sensitivity of the model with respect to several discretionary parameters.

1 Introduction

The distinction between the actions of individual agents and aggregate behaviour has been a central theme in macroeconomics at least since the work of Keynes, who in Keynes (1936) stated that:

> For although the amount of his own saving is unlikely to have any significant influence on his own income, the reactions of the amount of his consumption on the incomes of others makes it impossible for all individuals simultaneously to save any given sums. Every such attempt to save more by reducing consumption will so affect incomes that the attempt necessarily defeats itself. It is, of course, just as impossible for the community as a whole to save less than the amount of current investment, since the attempt to do so will necessarily

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†Email address: grasselli@math.mcmaster.ca. Partially supported by a Discovery Grant from the Natural Science and Engineer Research Council of Canada (NSERC) and the Institute for New Economic Thinking (INET) Grant INO13-00011. Hospitality of the Center for Financial Mathematics and Actuarial Research, University of Santa Barbara, where this work was completed, is also gratefully acknowledged.

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raise incomes to a level at which the sums which individuals choose to save add up to a figure exactly equal to the amount of investment.

Given the inherent challenges in assessing individual behaviour of a large number of agents, Keynesian economics tended to focus instead on direct modelling of aggregate variables, such as total savings and output. The difficulty with this approach is that it downplays the role of individual decision making, in particular in the face of uncertainty. On the opposite end of the spectrum, the predominant Dynamic Stochastic General Equilibrium (DSGE) models of contemporary macroeconomics advocate that all aggregate relationships need to be derived from individual decision-making agents, what is known as *microfoundations*. The problem with this position, however, is that, as a consequence of the celebrated Sonnenschein-Mantel-Debreu (SMD) theorem (see for example Mantel (1974)), the hypothesized properties of individual agents (namely inter-temporal utility maximizing) are in general not enough to guarantee that the resulting aggregate behaviour (namely general equilibrium) is stable. To circumvent this fundamental difficulty, DSGE models typically assume that each relevant sector of the economy consists of a single *representative agent*, thereby avoiding the aggregation problem associated the SMD theorem. Naturally, this simplification also throws away any possibility for emerging behaviour, as the aggregate and individual levels are automatically assumed to be identical, a weakness that has been widely seen to be a core reason for the poor performance of DSGE models during the recent crisis (see for example Kirman (2010)).

In recent years, several papers have attempted to extend DSGE models and incorporate heterogeneous agents, starting with the seminal contributions of Krusell and Smith (1998) for heterogeneous households and Khan and Thomas (2008) for heterogeneous firms. A recent survey of such models can be found in Ragot (2017), where it is explained that heterogeneity is introduced through a series of idiosyncratic shocks experienced by different agents. The shocks can represent, for example, different levels of employment income for households or different levels of capital productivity for firms, and are typically modelled by a finite-state Markov chain with constant transition probabilities. Beyond these exogenous shocks, agents are still considered to be identical with respect to their decision making. For example, in Krusell and Smith (1998) all households have the same utility function, whereas in Khan and Thomas (2008) all firms are profit optimizers. In other words, apart from suffering from technical problems of their own[1], these models do not address the lack of heterogeneity in behaviour that is common to all DSGE models as a consequence of the SMD results.

An alternative to both aggregate-level Keynesian and *representative*-agent-based DSGE models in macroeconomics consists of agent-based models (ABM), where agents are not constrained by utility maximizing behaviour and aggregation is not achieved through equilibrium. The literature on these models burgeoned since the 2007-08 crisis, and a recent assessment of the results, including a comparison with DSGE models, can be found, for example, in Fagiolo and Roventini (2016). A common objection to ABM is that they typically rely almost exclusively on numerical simulations, making them both computationally intense and difficult to interpret. This is particularly acute when an ABM, as it is often the case, has many underlying parameters. In the absence of a faster way to simulate the

[1] For example, strictly speaking, as observed in Ragot (2017), they are not DSGE models, since each realization of the sequence of shocks gives rise to a new agent. Several approximations are then used to "solve" the models in a way that resembles their motivating DSGE core, such as truncating the history of shocks to a fix number of past time steps.

model, parameter estimation, for example, can become prohibitively slow. One way to address this problem is to introduce semi-analytic approximations by way of mean-field interactions, as advocated for example in Gallegati and Kirman (2012).

The general mathematical framework for the application of mean-field (MF) approximations of this kind to economics can be found in Aoki (2002). An application to a specific model explaining business cycles fluctuations is presented in Di Guilmi et al. (2010) and Delli Gatti et al. (2012). The key feature of the approach consists of dividing the relevant sectors (say firms or households) into types according to some classification. Agents in the same type are then deemed to behave in a similar way (say with respect to investment or savings), so that one can keep track of averages (or other statistics) of the variables of interest, instead of their values for each individual agent. Crucially, the agents are assumed to make decisions also based on these averages, in what is called a mean-field interaction, rather than by direct interaction with other agents. Finally, agents are allowed to change type in a probabilistic manner, so that the time evolution of the distribution of agents is governed by the so-called master equation. This achieves considerable simplification by replacing the computation of quantities of interest for a large number of agents with a much smaller number of dynamical equations for averages for each type with the help of the corresponding master equation.

The accuracy of the approximation, nevertheless, depends on avoiding oversimplifications of the interactions between agents. In particular, as has been recently stressed in the literature on stock-flow consistent (SFC) models, economic agents are linked by credit and debt relationships that put constraints on both individual and aggregate balance sheets (see Caiani et al. (2016) for a recent ABM-SFC model), and these in turn need to be taken into account in the MF approximation. We illustrate this phenomenon in this paper using the models in Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017) as our starting points.

In Section 2 we introduce a stock-flow consistent agent-based model with two types of firms and two types of households. The firms can be either aggressive (type 1) or conservative (type 2), depending on how much their current level of investment reacts to past profits. Households, on the other hand, can be either non-investors (type 1) or investors (type 2), depending on whether or not they invest a portion of their savings in the stock market. Differently from Di Guilmi et al. (2010), Delli Gatti et al. (2012), Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), we assume in this paper that the probabilities for transitions between types are constant and exogenously given. As explained in the Appendix, this is because, as far as we can tell, the solution method for the master equation employed in these papers does not extend to the time-dependent, threshold-based transition probabilities they propose to use [2]. By contrast, we show in the Section 3 that, in the case of constant and exogenous transition probabilities, the so-called *ansatz* method, explained in detail in Aoki (2002) for the case of two types of agents, extends to the 2×2 case, namely when two types of agents in one sector (say firms) interact with two types of agents in another sector (say households).

Section 4 investigates both the ABM and its MF approximation. We first verify that the mean-field approximation gives rise to aggregate variables, such as equity prices and nominal output, that closely match the corresponding values obtained in simulations of the full agent-based model. Next we use the MF approximation to perform explorations of

[2]As pointed out by an anonymous referee, an alternative solution method for the mean-field approximation with time-varying transition rates has been recently proposed in Di Guilmi et al. (2017).

the parameter space that would be much slower with the ABM simulations. In particular, we investigate the behaviour of aggregate variables with respect to parameters that are difficult to estimate outside the model, such as the fraction of external financing that firms raise by issuing new debt as opposed to equity.

2 The model

We assume that the economy consists of an aggregate banking sector (henceforth referred to as "the bank"), N firms indexed by $n = 1, \ldots, N$, and M households indexed by $m = 1, \ldots, M$. The N firms collectively produce a total output Q_t at time t, which determines the total demand for labour and total wage bill as $L_t = Q_t/a$ and $W_t = cQ_t$, where a is the productivity per unit of labour and c is the labour cost per unit of output. As in Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), we ignore labor market dynamics by assuming that both c and a are constant. We next assume that the price of each unit of output is given by

$$p_t = \chi c, \tag{1}$$

where $\chi \geq 1$ is constant markup over unit cost. It then follows that the wage share of output is constant and given by

$$\omega = \frac{W_t}{pQ_t} = \frac{cQ_t}{\chi c Q_t} = \frac{1}{\chi}, \tag{2}$$

and consequently the profit share of output is also constant and given by

$$\pi = \frac{pQ_t - W_t}{pQ_t} = 1 - \omega = \frac{\chi - 1}{\chi} \tag{3}$$

We further assume that each household supplies L_t/M units of labour at time t, thereby receiving a wage rate[3]

$$\mathrm{w}_t = \frac{W_t}{M} = \frac{cQ_t}{M} = ca\frac{L_t}{M}, \tag{4}$$

which we assume to be the same for all households[4].

2.1 Balance Sheets

The balance sheets of each agent at time t are depicted in Table 1. Namely, firm n has assets consisting of capital with nominal value pk_t^n and liabilities consisting of net debt with nominal value b_t^n and e_t^n shares at average price $p_t^{e_n}$, leading to net worth equal to

$$v_t^n = pk_t^n - b_t^n - p_t^{e_n} e_t^n. \tag{5}$$

[3] Alternatively, we could follow Di Guilmi and Carvalho (2017) and assume that there are L_t households employed at time t, each supplying one unit of labour at a constant average wage rate $\mathrm{w} = ca$. The disadvantage of this approach is that the number of employed households fluctuates in time, creating a distinction among households in addition to the types introduced in Section 2.2.

[4] In Carvalho and Di Guilmi (2014), each household is subject to a further idiosyncratic shock to its wage rate. We do not pursue this approach here, as the only sources of randomness in our model are the transitions between types of firms and households with exogenous rates introduced in Section 2.3. Additional demand or supply shocks can be modelled separately.

Notice that, to simplify the notation, we treat net debt b_t^n as the difference between loans and deposits for firm n, which can therefore be positive or negative depending on whether firm n is a net borrower or lender (i.e depositor), respectively. Observe that we follow the accounting convention advocated in Godley and Lavoie (2007), namely that equity issued by firms should be treated as a financial liability booked at market value. This is done for consistency with national accounts, where equity held by households is treated as a financial asset for shareholders and also accounted at market value. Notice that the net worth in (5) resulting from this convention is typically much smaller than the more common corporate accounting concept of shareholder equity, which in our context corresponds to

$$\epsilon^n = pk_t^n - b_t^n, \tag{6}$$

that is, the "book value" of the difference between assets (physical capital and deposits) and debt liabilities (loans). The discrepancy between the market and book values of equity is captured by the valuation ratio or Tobin's q, which in our context reduces to

$$q_t^n = \frac{p_t^{e_n} e_t^n + b_t^n}{\epsilon^n + b_t^n} = \frac{pk_t^n - v_t^n}{pk_t^n}, \tag{7}$$

from which we can see that the net worth for firm n in (5) is positive if, and only if, its q-ratio is less than one, meaning that the market undervalues the firm.

Similarly, household m has assets consisting of e_t^m shares at average price $p_t^{e_m}$ and cash balances d_t^m deposited at the bank, leading to a net worth

$$v_t^m = p_t^{e_m} e_t^m + d_t^m. \tag{8}$$

Notice that we again treat d_t^m as the difference between deposits and loans for household m, which can therefore be positive of negative depending on whether household m is a net lender or borrower, respectively. The balance sheet of the bank accommodates the demands for loans and deposits across the economy. Accordingly, its assets consist of aggregate net borrowing by firms

$$B_t = \sum_{n=1}^{N} b_t^n, \tag{9}$$

plus cash reserves R_t, and its liabilities consist of aggregate net deposits of households

$$D_t = \sum_{m=1}^{M} d_t^m, \tag{10}$$

leading to a net worth of the form $V_t^b = B_t + R_t - D_t$.

Regarding equities, as we shall see below, we will assume a homogenous behaviour for firms with respect to dividend payments and share issuance and buyback. Based on this, we make the simplifying assumption that, instead of trading in shares for individual companies, investors buy and sell shares of an aggregated fund at a common price p_t^e, which in turn buys and sells shares from firms. The price p_t^e is then determined by an equilibrium condition for the supply and demand for equities under the constraint that

$$\sum_{n=1}^{N} e_t^n = E_t = \sum_{m=1}^{M} e_t^m. \tag{11}$$

Firm n

pk_t^n	b_t^n $p_t^{e_n} e_t^n$ v_t^n

Household m

$p_t^{e_m} e_t^m$ d_t^m	v_t^m

Bank

B_t	D_t
R_t	V_t^b

Table 1: Balance sheets at t.

2.2 Transactions and Aggregate Demand

We consider a demand-driven economy operating below maximum capacity, so that, given aggregate demand Q_t, firms adjust production according to

$$q_t^n = f_t^n Q_t, \quad f_t^n > 0, \quad \sum_{n=1}^{N} f_t^n = 1, \tag{12}$$

where the fractions f_t^n are known at time t. For example, we can have $f_t^n = f_0^n$ for a constant vector of allocations $(f_0^1, \cdots, f_0^n)$ or, alternatively, consider a preferential attachment rule of the form

$$f_t^n = \frac{k_t^n}{K_t}, \qquad K_t = \sum_{n=1}^{N} k_t^n, \tag{13}$$

that is to say, firms with larger capital at time t receive a larger share of demand[5]. Each firms is classified as either aggressive (type 1) or conservative (type 2). We assume that firm n decides on its investment at $t+1$ based on its previous type $z_t^n \in 1, 2$, gross profits $\pi p q_t^n$, production level pq_t^n (used as a proxy for capacity utilization) and debt b_t^n according to

$$i_{t+1}^n = \alpha_{z_t^n} \pi p q_t^n + \beta p q_t^n - \gamma b_t^n = (\alpha_{z_t^n} \pi + \beta) p q_t^n - \gamma b_t^n, \tag{14}$$

where α_z, β and γ denote the sensitivity of investment to gross profits, capacity utilization, and current level of debt, respectively. We assume that $\alpha_1 > \alpha_2$, that is to say, investment by aggressive firms is more sensitive to gross profits than for conservative ones. This in turn determines capital for firm n at time $t+1$ by

$$pk_{t+1}^n = i_{t+1}^n + (1-\delta) pk_t^n, \tag{15}$$

as well as aggregate capital $K_{t+1} = \sum_{n=1}^{N} k_{t+1}^n$. Observe, in particular, that the aggregate capital evolves as

$$p(K_{t+1} - K_t) = I_{t+1} - \delta p K_t, \tag{16}$$

where $I_{t+1} = \sum_{n=1}^{N} i_{t+1}^n$ denotes total investment. Aggregate demand Q_{t+1} is determined by equilibrium in the goods market once consumption by households is specified. Assuming that f_{t+1}^n is known at time $t+1$ (say constant or, alternatively equal to $f_{t+1}^n = k_{t+1}^n / K_{t+1}$), the share of production for firm n is again obtained as

$$q_{t+1}^n = f_{t+1}^n Q_{t+1}, \tag{17}$$

[5] In Di Guilmi and Carvalho (2017), this share is further subject to an idiosyncratic shock that redistributes demand among firms such that $E[q_t^n] = \frac{k_t^n}{K_t} Q_t$ and $\sum_{n=1}^{N} q_t^n = Q_t$. We do not pursue this approach either, as the only source of randomness is the transition between types of firms and households with exogenous rates introduced in Section 2.3. Additional demand or supply shocks can be modelled separately.

which in turn determines the gross profit for firm n as $\pi p q^n_{t+1}$. The amount of retained profits available to firm n to finance investment at time $t+1$ is then given by

$$a^n_{t+1} = \pi p q^n_{t+1} - r b^n_t - \delta p k^n_t - \delta^e p^e_t e^n_t, \tag{18}$$

where rb^n_t are interest charges on debt held at time t, $\delta p k^n_t$ are depreciation charges (otherwise known in accounting as consumption of fixed capital), and $\delta^e p^e e^n_t$ are dividends paid to shareholders according to a dividend yield δ^e, which we assume to be constant and equal for all firms.

The two classes of households correspond to non-investors (type 1), for whom $e^m_t = 0$, and investors (type 2), for whom $e^m_t > 0$. Accordingly, the disposable income to be received by household m at time $t+1$ consists of

$$y^m_{t+1} = \mathrm{w}_{t+1} + r d^m_t + \delta^e p^e_t e^m_t. \tag{19}$$

where $\mathrm{w}_{t+1} = (1-\pi)pQ_{t+1}/M$ is the effective wage rate obtained in (4), rd^m_t is interest paid on deposits d^m_t held at time t and the last term represents dividends paid to household m, which we assume to be a fraction e^m_t/E_t of the total amount of dividends $\delta^e p^e_t E_t$ paid by firms. In other words, we assume that household m receives dividends in proportion to their equity holdings before rebalancing their portfolio, and in particular before changing type, at time $t+1$.

Household m then decides on its consumption at time $t+1$ based on its previous state $z^m_t \in 1, 2$, current disposable income y^m_{t+1}, and previous wealth $v^m_t = d^m_t + p^e_t e^m_t$ according to

$$c^m_{t+1} = (1 - s^y_{z^m_t}) y^m_{t+1} + (1 - s^v_{z^m_t}) v^m_t, \tag{20}$$

where $s^y_z, s^w_z \in [0, 1]$ are the saving rates from income and wealth, respectively. We assume that $s^y_1 \leq s^y_2$ and $s^v_1 \leq << s^v_2$, so that investors save a higher proportion of both income and wealth than non-investors.

Nominal aggregate demand at time $t+1$ is then given by

$$pQ_{t+1} = I_{t+1} + C_{t+1}, \tag{21}$$

with

$$I_{t+1} = \sum_{n=1}^{N} i^n_{t+1} = \pi p(\alpha_1 Q^1_t + \alpha_2 Q^2_t) + \beta p Q_t - \gamma B_t, \tag{22}$$

and

$$\begin{aligned} C_{t+1} = \sum_{m=1}^{M} c^m_{t+1} &= (1 - s^y_1)\left[(1-\pi)pQ_{t+1} m^1_t + r D^1_t\right] + (1 - s^v_1) D^1_t \\ &+ (1 - s^y_2)\left[(1-\pi)pQ_{t+1} m^2_t + r D^2_t + \delta^e p^e_t E_t\right] + (1 - s^v_2)(D^2_t + p^e_t E_t). \end{aligned} \tag{23}$$

where we used the notation m^z_t for the proportion of households of type z at time t and also introduced the class aggregates

$$Q^z_t = \sum_{\{n: z^n_t = z\}} q^n_t, \qquad D^z_t = \sum_{\{m: z^m_t = z\}} d^m_t. \tag{24}$$

Substituting (22) and (23) into (21) we find that aggregate demand at $t+1$ can be calculated from quantities known at time t as follows:

$$pQ_{t+1} = \frac{F_t}{1-(1-\pi)[(1-s_1^y)m_t^1 + (1-s_2^y)m_t^2]}, \tag{25}$$

where

$$\begin{aligned} F_t =& \pi p(\alpha_1 Q_t^1 + \alpha_2 Q_t^2) + \beta p Q_t - \gamma B_t + (1-s_1^y) r D_t^1 + (1-s_1^v) D_t^1 \\ &+ (1-s_2^y)(r D_t^2 + \delta^e p_t^e E_t) + (1-s_2^v)(D_t^2 + p_t^e E_t). \end{aligned} \tag{26}$$

2.3 Transitions

We assume that, after making its investment decision for time $t+1$, each firm n undergoes a transition to determine its new type according to the conditional probabilities

$$P_{ij}^f(t) := \text{Prob}\left(z_{t+1}^n = j | z_t^n = i\right) = \begin{pmatrix} 1-\mu^f & \mu^f \\ \lambda^f & 1-\lambda^f \end{pmatrix} \tag{27}$$

In words, each of the N_t^1 firms in state $z=1$ (aggressive) at time t decides to transition to state $z=2$ (conservative) at time $t+1$ with probability μ^f. Similarly, each of the $N-N_t^1$ firms in state 2 at t decides to transition to state 1 with probability λ^f. Here μ^f and λ^f are constant parameters specified exogenously.

Similarly, after making a consumption decision for time $t+1$, each household m undergoes a transition to determine its new type according to the conditional probabilities

$$P_{ij}^h(t) := \text{Prob}\left(z_{t+1}^m = j | z_t^m = i\right) = \begin{pmatrix} 1-\mu^h & \mu^h \\ \lambda^h & 1-\lambda^h \end{pmatrix} \tag{28}$$

That is, each of the M_t^1 households in state $z=1$ (non-investor) at time t decides to transition to state $z=2$ (investor) at time $t+1$ with probability μ^h. Similarly, each of the $M-M_t^1$ households in state 2 at t decides to transition to state 1 with probability λ^h. As with the rates for firms, μ^h and λ^h are constant parameters specified exogenously.

This specification of transition probabilities is a significant departure from the models in Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), where the transition probabilities are specified as functions of balance sheet variables for each agent. As a result, our model has both significantly different behavioural assumptions and analytical properties.

More specifically, whereas we classify firms as either aggressive or conservative depending on some exogenously determined propensity to invest, Di Guilmi and Carvalho (2017) classify them as speculative and hedge firms, depending on whether or not they need to borrow in order to finance their investment. From a behavioural point of view, our assumption means that the balance sheet of firm n at time t affects its investment decision only indirectly through the debt level b_t^n in (14), whereas the elasticity $\alpha_{z_t^n}$ depends on an independently specified random variable, namely its type at time t, for example having to do with the "animal spirits" of the managers of the firm at the time. In other words, in our model, two firms with identical balance sheets and facing the same demand q_t can still make different investment decisions according to (14) provided one is aggressive and the other is conservative at the time. By contrast, the type, and consequently the investment behaviour of a firm in Di Guilmi and Carvalho (2017) is entirely determined by its balance sheet: two firms with identical balance sheets and facing the same demand will necessarily make the same investment decision. Similar remarks apply to the classification of households in Carvalho and Di Guilmi (2014) as borrowing and non-borrowing, instead of investor and non-investor in our model.

From an analytical point of view, as mentioned in the Appendix, the transition probabilities adopted in Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), which were themselves adapted from earlier work in Di Guilmi et al. (2010), do not satisfy the conditions necessary to derive the approximation of the Master Equation described in Section 3.2, making it difficult to justify the use of the mean field approximation for their models.

2.4 Flow of funds

When net investment $i^n_{t+1} - \delta p k^n_t$ for firm n exceeds its retained profits a^n_{t+1}, the difference needs to be financed by new borrowing from the banking sector or issuance of new shares. Conversely, if the amount of investment is lower than retained profits, then the excess funds can be used to pay down outstanding debt or to buy back shares. Following Carvalho and Di Guilmi (2014), we assume that firm n raises external funds according to the proportions

$$b^n_{t+1} - b^n_t = \varpi(i^n_{t+1} - \delta p k^n_t - a^n_{t+1}) \tag{29}$$

$$p^e_{t+1}(e^n_{t+1} - e^n_t) = (1-\varpi)(i^n_{t+1} - \delta p k^n_t - a^n_{t+1}), \tag{30}$$

where $0 \leq \varpi \leq 1$ is a constant common to all firms. The debt b^n_{t+1} held by firm n at time $t+1$ can therefore be determined by (29), whereas the number of shares e^n_{t+1} outstanding for firm n at time $t+1$ depends on the equity price p^e_{t+1} to be determined by equilibrium below. The total supply of equities at time $t+1$ is given by

$$p^e_{t+1} E_{t+1} = p^e_{t+1} E_t + (1-\varpi)(I_{t+1} - \delta p K_t - A_{t+1}), \tag{31}$$

where I_{t+1} is defined in (22) and

$$A_{t+1} = \sum_{n=1}^{N} a^n_{t+1} = \pi p Q_{t+1} - rB_t - \delta p K_t - \delta^e p^e_t E_t, \tag{32}$$

denotes the total amount of retained profits, or in other words, total savings for the firm sector.

On the other hand, savings for household m at time $t+1$ are given by

$$s^m_{t+1} = y^m_{t+1} - c^m_{t+1}, \tag{33}$$

which, as we have seen, only depends on quantities that are known at time t, including its type z^m_t and aggregate demand Q_{t+1} given by (25). Accordingly, total savings for the household sector are given by

$$S_{t+1} = \sum_{m=1}^{M} s^m_{t+1} = \sum_{m=1}^{M} \left(y^m_{t+1} - c^m_{t+1}\right) = (1-\pi)pQ_{t+1} + rD_t + \delta^e p^e_t E_t - C_{t+1} \tag{34}$$

The change in wealth for household m is then given by savings plus capital gains, that is,

$$v^m_{t+1} = v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t)e^m_t. \tag{35}$$

This wealth at time $t+1$ is then allocated into deposits and equities according to the new type z_{t+1}^m. Namely, we assume that the demand for equities for household m is given by

$$p_{t+1}^e e_{t+1}^m = \varphi v_{t+1}^m (z_{t+1}^m - 1) = \begin{cases} 0 & \text{if } z_{t+1}^m = 1 \\ \varphi v_{t+1}^m & \text{if } z_{t+1}^m = 2, \end{cases} \tag{36}$$

where φ is a constant common to all households and we recall that $z_{t+1}^m = 2$ if household m is an investor (type 2) and $z_{t+1}^m = 1$ otherwise (type 1). The demand for deposits for household m is then given by the residual

$$d_{t+1}^m = v_{t+1}^m - p_{t+1}^e e_{t+1}^m = \begin{cases} v_{t+1}^m & \text{if } z_{t+1}^m = 1 \\ (1-\varphi) v_{t+1}^m & \text{if } z_{t+1}^m = 2. \end{cases} \tag{37}$$

Accordingly, total demand for equities by households is given by

$$\begin{aligned} p_{t+1}^e E_{t+1} &= \varphi \left(\sum_{\{m: z_{t+1}^m = 2\}} v_{t+1}^m \right) = \varphi \left(\sum_{\{m: z_{t+1}^m = 2\}} v_t^m + s_{t+1}^m + (p_{t+1}^e - p_t^e) e_t^m \right) \\ &= \varphi \left(\sum_{\{m: z_{t+1}^m = 2\}} d_t^m + p_t^e e_t^m + s_{t+1}^m + (p_{t+1}^e - p_t^e) e_t^m \right) \\ &= \varphi \left(D_t^{2,t+1} + S_{t+1}^2 + p_{t+1}^e E_t^{2,t+1} \right), \end{aligned} \tag{38}$$

where we introduced the class aggregates

$$\begin{aligned} D_t^{2,t+1} &= \sum_{\{m: z_{t+1}^m = 2\}} d_t^m, \\ S_{t+1}^2 &= \sum_{\{m: z_{t+1}^m = 2\}} s_{t+1}^m, \\ E_t^{2,t+1} &= \sum_{\{m: z_{t+1}^m = 2\}} e_t^m. \end{aligned}$$

In these expressions, notice that the upper time index refers to the time in which the type z_{t+1}^m is evaluated, whereas the lower time index refers to the time in which the summands are evaluated. In words, $D_t^{2,t+1}$ is the sum of deposits held at time t by households that are of type 2 at time $t+1$. When the upper and lower time indices coincide, we suppress the upper index, in accordance with the notation introduced in (24).

Equating the total supply of equities in (31) with the total demand for equities in (38) leads to an equilibrium equity price of the form

$$p_{t+1}^e = \frac{\varphi \left(D_t^{2,t+1} + S_{t+1}^2 \right) - (1-\varpi)\left(I_{t+1} - \delta p K_t - A_{t+1} \right)}{E_t - \varphi E_t^{2,t+1}}. \tag{39}$$

This can then be used in (30) to obtain the number of shares e_{t+1}^n outstanding for firm n at time $t+1$, and consequently the total number of shares $E_{t+1} = \sum_{n=1}^N e_{t+1}^n$.

We can now perform two stock-flow consistency checks by calculating the savings for the bank at $t+1$ as the change in its net worth, namely

$$S^b_{t+1} = V^b_{t+1} - V^b_t = (B_{t+1} - B_t) - (D_{t+1} - D_t). \tag{40}$$

Observe first that it follows from (29) that

$$B_{t+1} - B_t = \sum_{n=1}^{N} \varpi(i^n_{t+1} - \delta p k^n_t - a^n_{t+1}) = \varpi(I_{t+1} - \delta p K_t - A_{t+1}). \tag{41}$$

Next, we compute the total amount of deposits at time $t+1$ as

$$\begin{aligned}
D_{t+1} = \sum_{m=1}^{M} d^m_{t+1} &= \sum_{\{m:z^m_{t+1}=1\}} v^m_{t+1} + \sum_{\{m:z^m_{t+1}=2\}} (1-\varphi) v^m_{t+1} \\
&= \sum_{\{m:z^m_{t+1}=1\}} v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t) e^m_t \\
&\quad + (1-\varphi)\left(\sum_{\{m:z^m_{t+1}=2\}} v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t) e^m_t \right) \\
&= \sum_{\{m:z^m_{t+1}=1\}} d^m_t + s^m_{t+1} + p^e_{t+1} e^m_t \\
&\quad + (1-\varphi)\left(\sum_{\{m:z^m_{t+1}=2\}} d^m_t + s^m_{t+1} + p^e_{t+1} e^m_t \right) \\
&= D^{1,t+1}_t + S^1_{t+1} + p^e_{t+1} E^{1,t+1}_t + (1-\varphi)\left(D^{2,t+1}_t + S^2_{t+1} + p^e_{t+1} E^{2,t+1}_t \right) \\
&= D_t + S_{t+1} + p^e_{t+1} E_t - \varphi\left(D^{2,t+1}_t + S^2_{t+1} + p^e_{t+1} E^{2,t+1}_t \right) \\
&= D_t + S_{t+1} + p^e_{t+1} E_t - p^e_{t+1} E_{t+1} \\
&= D_t + S_{t+1} - (1-\varpi)(I_{t+1} - \delta p K_t - A_{t+1}).
\end{aligned} \tag{42}$$

where we used $v^m_t = d^m_t + p^e_t e^m_t$ to move from the second to the third line above, in addition to (38) and (31) in the last two lines. Substituting (41) and (42) in (40) gives

$$S^b_{t+1} + S_{t+1} + A_{t+1} = I_{t+1} - \delta p K_t, \tag{43}$$

which confirms that net investment at time $t+1$ equals the total savings across the three sectors in the economy. Furthermore, using the expressions (32) and (34) we find that

$$\begin{aligned}
S^b_{t+1} =& I_{t+1} - \delta p K_t - (\pi p Q_{t+1} - r B_t - \delta p K_t - \delta^e p^e_t E_t) \\
& - [(1-\pi) p Q_{t+1} + r D_t + \delta^e p^e_t E_t - C_{t+1}] \\
=& I_{t+1} + C_{t+1} - p Q_{t+1} + r B_t - r D_t = r(B_t - D_t),
\end{aligned} \tag{44}$$

confirming that profits for the bank consists of the interest differential between loans and deposits.

2.5 Special Cases

When all households are of the same type z, aggregate disposable income becomes

$$Y_{t+1} = (1-\pi)pQ_{t+1} + rD_t + \delta^e p_t^e E_t \mathbf{1}_{\{z=2\}}, \tag{45}$$

where $\mathbf{1}_{\{z=2\}} = 1$ is all households are investors and zero otherwise, and consumption is given by

$$C_{t+1} = (1-s_z^y)Y_{t+1} + (1-s_z^v)V_t, \tag{46}$$

where $V_t = D_t + p_t^e E_t \mathbf{1}_{\{z=2\}}$. In this case, aggregate demand is given by

$$pQ_{t+1} = \frac{I_{t+1} + (1-s_z^y)\left(rD_t + \delta^e p_t^e E_t \mathbf{1}_{\{z=2\}}\right) + (1-s_z^v)V_t}{1-(1-\pi)(1-s_z^y)}, \tag{47}$$

where I_{t+1} is still given by (22). If all households are non-investors, then (47) reduces to equation (19) in Di Guilmi and Carvalho (2017)[6]. In this case, we should also impose that $\varpi = 1$, since there is no active equity market where firms can raise funds. On the other hand, if all households are investors, that the equity price in (39) reduces to

$$p_{t+1}^e = \frac{\varphi\left(D_t + S_{t+1}\right) - (1-\varpi)\left(I_{t+1} - \delta pK_t - A_{t+1}\right)}{(1-\varphi)E_t}, \tag{48}$$

which coincides with equation (35) in Carvalho and Di Guilmi (2014)[7] with φ as a constant proportion of wealth invested in equities instead of the variable proportion adopted in their equation (19).

Conversely, when all firms are of the same type, aggregate investment becomes

$$I_{t+1} = (\pi\alpha + \beta)pQ_t - \gamma B_t, \tag{49}$$

which reduces to equation (7) in Carvalho and Di Guilmi (2014) apart from obvious modifications[8].

3 Mean-Field Approximation

The model of the previous section can be readily implemented as an agent-based model (ABM) for reasonably large numbers of firms and households. Because of the probabilistic nature of the transitions between types of agents, the effects of the different model parameters on the asymptotic properties of the model are not immediately clear, and algebraic manipulation of the discrete-time equations governing its dynamic evolution

[6] With the extra assumption that households do not receive any interest on deposits, as it is implicitly assumed in Di Guilmi and Carvalho (2017)

[7] Notice that our definition of retained profits A_t differ from that in Carvalho and Di Guilmi (2014) in two ways. First, we subtract depreciation costs from gross profits, as it is commonly done in accounting, while at the same time subtracting the same amount from gross investment. Secondly, we assume that distributed profits take the form of a constant dividend *yield* δ^e, rather than a constant dividend *payout ratio* Θ, which avoids the anomaly of paying out negative dividends when earnings are negative.

[8] Namely, the effect of debt on investment is not considered in Carvalho and Di Guilmi (2014), corresponding to $\gamma = 0$ in our setting. Conversely, we do not consider either a desired capacity utilization or the effect of stock valuation on investment, corresponding to setting their constants u^d and ε to zero. Redefining the roles of α and β completes the identification between our equation (49) and their equation (7)

proves to be both tedious and challenging. The purpose of this section is to present a mean-field approximation approach along the lines proposed in Di Guilmi et al. (2010) and followed in Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), as an alternative to large scale numerical simulations of the discrete-time agent-based model.

3.1 Discrete-time mean-field dynamics

The first ingredient of the approach consists in homogenizing the populations of firms and households of a given type by expressing the discrete-time model in terms of "mean-field" variables that are common to all agents of the same type. Notice that this is very different from the representative agent framework mentioned in Section 1 in connection with DSGE models. Our use of mean-field values over a group of agents of the same type is done purely for computational convenience, whereas the use of representative agents is essentially the only way to guarantee the stability of equilibrium in DSGE models as a consequence of the Sonnenschein-Mantel-Debreu theorem (see for example Kirman (1992)).

We denote the average values of a variable x for agents of type z at time t by $\overline{x}^z_t$. Its time evolution requires two steps: we first compute the deterministic value $\widetilde{x}^z_{t+1}$ *before* agents change type at time $t+1$ and then calculate the new mean-field value $\overline{x}^z_{t+1}$ taking into account the changes in type. This is necessary because agents carry their balance sheet items with them when they change type, and consequently both the aggregate and average values of a variable for agents of type z change when agents change type[9]. Accordingly, since the average number of firms changing from type 1 to type 2 is μN^1_t and the average number of firms changing from type 2 to type 1 is $\lambda(N - N^1_t)$, we set the mean-field values of x for firms of type z *after* a change of type at time $t+1$ as

$$\overline{x}^1_{t+1} = \frac{(1-\mu)N^1_t\widetilde{x}^1_{t+1} + \lambda(N - N^1_t)\widetilde{x}^2_{t+1}}{N^1_{t+1}} \tag{50}$$

and

$$\overline{x}^2_{t+1} = \frac{\mu N^1_t\widetilde{x}^1_{t+1} + (1-\lambda)(N - N^1_t)\widetilde{x}^2_{t+1}}{N - N^1_{t+1}} \tag{51}$$

In this way, we find that

$$X_{t+1} = N^1_{t+1}\overline{x}^1_{t+1} + (N - N^1_{t+1})\overline{x}^2_{t+1} = N^1_t\widetilde{x}^1_{t+1} + (N - N^1_t)\widetilde{x}^2_{t+1} = \widetilde{X}_{t+1}, \tag{52}$$

so that the aggregate values for the variable x across the entire economy are the same before and after a change of type at time $t+1$, as they should be. Similar expressions hold for mean-field variables for households, with M^z_t and M^z_{t+1} replacing N^z_t and N^z_{t+1}.

In the context of the present model, suppose that, at time t, we are given the total production Q_t, the mean-field variables $\overline{k}^z_t$, $\overline{b}^z_t$, $\overline{e}^z_t$ and the number of firms N^z_t of type $z = 1, 2$, as well as the mean-field variable $\overline{d}^z_t$ and the number of M^z_t of type $z = 1, 2$. We then compute the mean-field production for each type as the analogues of (12), that is,

$$\overline{q}^z_t = \overline{f}^z_t Q_t, \quad z = 1, 2, \tag{53}$$

[9] Rebalancing after a change of type does not seem to be considered in either Carvalho and Di Guilmi (2014) and Di Guilmi and Carvalho (2017), even though this leads to puzzling behaviour in aggregate variables. For example, letting $p\overline{k}^z_{t+1} = \overline{i}^z_{t+1} + (1-\delta)p\overline{k}^z_t$ and ignoring rebalancing, it is easy to see that $pK_{t+1} \neq I_{t+1} + (1-\delta)pK_t$, where $K_{t+1} = N^1_{t+1}\overline{k}^1_{t+1} + (N - N^1_{t+1})\overline{k}^2_{t+1}$, $I_{t+1} = N^1_{t+1}\overline{i}^1_{t+1} + (N - N^1_{t+1})\overline{i}^2_{t+1}$, and $K_t = N^1_t\overline{k}^1_t + (N - N^1_t)\overline{k}^2_t$.

where $\overline{f}_t^z > 0$ are known at time t and satisfy $N_t^1\overline{f}_t^1 + (N - N_t^1)\overline{f}_t^2 = 1$. For example, they can be set to $\overline{f}_t^z = \overline{f}_0^1/N_t^1$ for a constant vector $(\overline{f}_0^1, \overline{f}_0^2)$ satisfying $\overline{f}_0^1 + \overline{f}_0^2 = 1$ or, alternatively, be proportional to the mean-field capital for each type, that is $\overline{f}_t^z = \overline{k}_t^z/K_t$, where $K_t = N_t^1\overline{k}_t^1 + (N - N_t^1)\overline{k}_t^2$ is the aggregate capital. The mean-field investment demand before firms change type at $t+1$ is then given by the analogue of (14), namely

$$\widetilde{i}_{t+1}^z = (\alpha_z\pi + \beta)p\overline{q}_t^z - \gamma\overline{b}_t^z, \quad z = 1, 2, \tag{54}$$

Accordingly, the mean-field capital $\widetilde{k}_{t+1}^z$ before a change in type at $t+1$ is given by the analogue of (15), that is,

$$p\widetilde{k}_{t+1}^z = \widetilde{i}_{t+1}^z + (1-\delta)p\overline{k}_t^z, \quad z = 1, 2. \tag{55}$$

Once aggregate demand $\widetilde{Q}_{t+1}$ is determined by equilibrium in the goods market, the mean-field productions before firms change type at $t+1$ can then be obtained as

$$\widetilde{q}_{t+1}^z = \widetilde{f}_{t+1}^z\widetilde{Q}_{t+1}, \tag{56}$$

where $\widetilde{f}_{t+1}^z$ is known at $t+1$ before firms change type (say, it is given by $\widetilde{f}_{t+1}^z = \widetilde{k}_{t+1}^z/\widetilde{K}_{t+1}$, where $\widetilde{K}_{t+1} = N_t^1\widetilde{k}_{t+1}^1 + (N - N_t^1)\widetilde{k}_{t+1}^2$ denotes aggregate capital in the economy before firms change type at $t+1$). After paying $r\overline{b}_t^z$ as interest charges on mean-field debt, $\delta p\overline{k}_t^z$ as depreciation costs for the mean-field capital, and dividends $\delta^e p_t^e\overline{e}_t^z$, all based on holdings at time t, the mean-field retained profits before changing type at $t+1$ are calculated as

$$\widetilde{a}_{t+1}^z = \pi p\widetilde{q}_{t+1}^z - r\overline{b}_t^z - \delta p\overline{k}_t^z - \delta^e p_t^e\overline{e}_t^z, \tag{57}$$

so that aggregate retained profits are given by

$$\widetilde{A}_{t+1} = \pi p\widetilde{Q}_{t+1} - rB_t - \delta pK_t - \delta^e p_t^e E_t \tag{58}$$

As in the ABM model, net investment in excess of retained profits needs to be financed externally by new debt and share issuance as follows

$$\widetilde{b}_{t+1}^z - \overline{b}_t^z = \varpi(\widetilde{i}_{t+1}^z - \delta p\overline{k}_t^z - \widetilde{a}_{t+1}^z) \tag{59}$$

$$p_{t+1}^e(\widetilde{e}_{t+1}^z - \overline{e}_t^z) = (1-\varpi)(\widetilde{i}_{t+1}^z - \delta p\overline{k}_t^z - \widetilde{a}_{t+1}^z). \tag{60}$$

We therefore have that the total supply of equities offered by firms satisfy

$$p_{t+1}^e\widetilde{E}_{t+1} = (1-\varpi)(\widetilde{I}_{t+1} - \delta p\widetilde{K}_t - \widetilde{A}_{t+1}) + p_{t+1}^e\widetilde{E}_t \tag{61}$$

Moving to households, the mean-field disposable incomes for types $z = 1, 2$ before a change in type at time $t+1$ are given by

$$\widetilde{y}_{t+1}^1 = \mathrm{w}_{t+1} + r\overline{d}_t^1 \tag{62}$$

$$\widetilde{y}_{t+1}^2 = \mathrm{w}_{t+1} + r\overline{d}_t^2 + \delta^e p_t^e\frac{E_t}{(M - M_t^1)}, \tag{63}$$

where $\mathrm{w}_{t+1} = (1-\pi)p\widetilde{Q}_{t+1}/M$. In other words, an equal fraction $(M - M_t^1)^{-1}$ of distributed profits $\delta^e p_t^e E_t$ is paid to each of the $(M - M_t^1)$ households of type 2 before a change of type at time $t+1$. Here $\widetilde{Q}_{t+1} = N_t^1\widetilde{q}_{t+1}^1 + (N - N_t^1)\widetilde{q}_{t+1}^2$.

The mean-field consumptions before a change in type at time $t+1$ is then

$$\widetilde{c}^1_{t+1} = (1-s^y_1)\left(\mathrm{w}_{t+1} + r\overline{d}^1_t\right) + (1-s^v_1)\overline{d}^1_t \tag{64}$$

$$\widetilde{c}^2_{t+1} = (1-s^y_2)\left(\mathrm{w}_{t+1} + r\overline{d}^2_t + \delta^e p^e_t \frac{E_t}{(M-M^1_t)}\right) + (1-s^v_2)\left(\overline{d}^2_t + p^e_t \frac{E_t}{(M-M^1_t)}\right). \tag{65}$$

We therefore have that aggregate demand at time $t+1$ before the change of type for firms and households is given by

$$p\widetilde{Q}_{t+1} = \widetilde{I}_{t+1} + \widetilde{C}_{t+1}, \tag{66}$$

with

$$\widetilde{I}_{t+1} = N^1_t \widetilde{i}^1_{t+1} + (N-N^1_t)\widetilde{i}^2_{t+1} = \pi p(\alpha_1 Q^1_t + \alpha_2 Q^2_t) + \beta p Q_t - \gamma B_t, \tag{67}$$

and

$$\begin{aligned}\widetilde{C}_{t+1} &= M^1_t \widetilde{c}^1_{t+1} + (M-M^1_t)\widetilde{c}^2_{t+1} \\ &= \frac{(1-\pi)p\widetilde{Q}_{t+1}}{M}\left[(1-s^y_1)M^1_t + (1-s^y_2)(M-M^1_t)\right] + (1-s^y_1)rD^1_t \\ &\quad + (1-s^y_2)\left(rD^2_t + \delta^e p^e_t E_t\right) + (1-s^v_1)D^1_t + (1-s^v_2)(D^2_t + p^e_t E_t),\end{aligned} \tag{68}$$

where we used the aggregate variables

$$Q^z_t = N^z_t \overline{q}^z_t, \quad D^z_t = M^z_t \overline{d}^z_t. \tag{69}$$

Substituting (68) into (66) we find that aggregate demand before a change of type at $t+1$ can be calculated from quantities known at time t as follows:

$$p\widetilde{Q}_{t+1} = \frac{F_t}{1-(1-\pi)[(1-s^y_1)m^1_t + (1-s^y_2)m^2_t]}, \tag{70}$$

where

$$\begin{aligned}F_t =& \pi p(\alpha_1 Q^1_t + \alpha_2 Q^2_t) + \beta p Q_t - \gamma B_t + (1-s^y_1)rD^1_t + (1-s^y_2)(rD^2_t + \delta^e p^e_t E_t) \\ &+ (1-s^v_1)D^1_t + (1-s^v_2)(D^2_t + p^e_t E_t).\end{aligned} \tag{71}$$

The mean-field savings for the two types of households before changing type at time $t+1$ are then given by

$$\widetilde{s}^z_{t+1} = \widetilde{y}^z_{t+1} - \widetilde{c}^z_{t+1}, \tag{72}$$

which, as in the ABM model, only depends on quantities known at time t. Total savings for the household sector before a change of type at $t+1$ are then given by

$$\widetilde{S}_{t+1} = M^1_t \widetilde{s}^1_{t+1} + (M-M^1_t)\widetilde{s}^2_{t+1} = (1-\pi)p\widetilde{Q}_{t+1} + rD_t + \delta^e p^e_t E_t - \widetilde{C}_{t+1}. \tag{73}$$

The mean-field wealth for the two types of households before changing type at $t+1$ are given by wealth at time t, plus savings, plus capital gains, that is

$$\widetilde{v}^1_{t+1} = \overline{v}^1_t + \widetilde{s}^1_{t+1} \tag{74}$$

$$\widetilde{v}^2_{t+1} = \overline{v}^2_t + \widetilde{s}^2_{t+1} + (p^e_{t+1} - p^e_t)\frac{E_t}{(M-M^1_t)} \tag{75}$$

Having computed $\widetilde{v}^z_{t+1}$ before a change of type, we let households change type at time $t+1$ as described in 3.2 and calculate the new mean-field values $\overline{v}^z_{t+1}$ according to the expressions (50) and (51) (suitably modified for M^z_t instead of N^z_t). The wealth for each type of agent is then reallocated into deposits and equities according to the new type at time $t+1$ as follows

$$\overline{d}^1_{t+1} = \overline{v}^1_{t+1} \tag{76}$$

$$\overline{d}^2_{t+1} = (1-\varphi)\overline{v}^2_{t+1} \tag{77}$$

$$p^e_{t+1}\frac{E_{t+1}}{(M-M^1_{t+1})} = \varphi\overline{v}^2_{t+1} \tag{78}$$

Accordingly, total demand for equities by households is given by

$$p^e_{t+1}E_{t+1} = \varphi(M-M^1_{t+1})\overline{v}^2_{t+1} = \varphi\left(D^{2,t+1}_t + S^2_{t+1} + p^e_{t+1}E^{2,t+1}_t\right), \tag{79}$$

where the mean-field analogues of the class aggregates introduced in (38) are

$$\begin{aligned} D^{2,t+1}_t &= \mu M^1_t \overline{d}^1_t + (1-\lambda)(M-M^1_t)\overline{d}^2_t \\ S^2_{t+1} &= \mu M^1_t \widetilde{s}^1_{t+1} + (1-\lambda)(M-M^1_t)\widetilde{s}^2_{t+1} \\ E^{2,t+1}_t &= (1-\lambda)E_t \end{aligned}$$

The interpretation of the upper and lower indices here is the same as before. For example, $D^{2,t+1}_t$ corresponds to deposits held at time t by households that are of type 2 at time $t+1$. Equating the total supply of equities (61) with the total demand (79) we find the equilibrium equity price at time $t+1$ given by

$$p^e_{t+1} = \frac{\varphi\left(D^{2,t+1}_t + S^2_{t+1}\right) - (1-\varpi)\left(I_{t+1} - \delta p K_t - A_{t+1}\right)}{[1-\varphi(1-\lambda)]E_t}, \tag{80}$$

from which we can calculate $\widetilde{e}^z_{t+1}$ in (60). Finally, having computed $\widetilde{k}^z_{t+1}$, $\widetilde{e}^z_{t+1}$ and $\widetilde{b}^z_{t+1}$, we calculate the new mean-field values $\overline{k}^z_{t+1}$, $\overline{b}^z_{t+1}$, and $\overline{e}^z_{t+1}$ according to the expressions (50) and (51) and verify that

$$p(K_{t+1} - K_t) = I_{t+1} - \delta p K_t, \tag{81}$$

$$B_{t+1} - B_t = \varpi(I_{t+1} - \delta p K_t - A_{t+1}), \tag{82}$$

$$p^e_{t+1}(E_{t+1} - E_t) = (1-\varpi)(I_{t+1} - \delta p K_t - A_{t+1}). \tag{83}$$

The same stock-flow consistency checks that we performed for the ABM model can now be done here. Observe that

$$\begin{aligned} D_{t+1} =& M^1_{t+1}\overline{d}^1_{t+1} + (M-M^1_{t+1})\overline{d}^2_{t+1} \\ =& M^1_{t+1}\overline{v}^1_{t+1} + (1-\varphi)(M-M^1_{t+1})\overline{v}^2_{t+1} \\ =& (1-\mu)M^1_t\widetilde{v}^1_{t+1} + \lambda(M-M^1_t)\widetilde{v}^2_{t+1} + \left[\mu M^1_t\widetilde{v}^1_{t+1} + (1-\lambda)(M-M^1_t)\widetilde{v}^2_{t+1}\right] - \varphi(M-M^1_{t+1})\overline{v}^2_{t+1} \\ =& M^1_t\left(\overline{d}^1_t + \widetilde{s}^1_{t+1}\right) + (M-M^1_t)\left(\overline{d}^2_t + \widetilde{s}^2_{t+1} + p^e_{t+1}\frac{E_t}{(M-M^1_t)}\right) - \varphi(M-M^1_{t+1})\overline{v}^2_{t+1} \\ =& M^1_t\left(\overline{d}^1_t + \widetilde{s}^1_{t+1}\right) + (M-M^1_t)\left(\overline{d}^2_t + \widetilde{s}^2_{t+1}\right) + p^e_{t+1}E_t - \varphi\left(D^{2,t+1}_t + S^2_{t+1} + p^e_{t+1}E^{2,t+1}_t\right) \\ =& D_t + \widetilde{S}_{t+1} + p^e_{t+1}E_t - p^e_{t+1}E_{t+1} \end{aligned}$$

where we have used (79). Using (83), we conclude that

$$D_{t+1} - D_t = \widetilde{S}_{t+1} - (1-\varpi)(I_{t+1} - \delta p K_t - A_{t+1}). \tag{84}$$

Using this and (82) we find that

$$S^b_{t+1} + \widetilde{S}_{t+1} + A_{t+1} = (B_{t+1} - B_t) - (D_{t+1} - D_t) + S_{t+1} + A_{t+1} = I_{t+1} - \delta p K_t, \tag{85}$$

so that total savings across all sectors of the economy equals net investment. Finally, inserting using (66), (73), and (58) in (85) and using the fact that, by construction, $X_{t+1} = \widetilde{X}_{t+1}$ for aggregate quantities, we find that

$$\begin{aligned} S^b_{t+1} =& I_{t+1} - \delta p K_t - [(1-\pi)p\widetilde{Q}_{t+1} + rD_t + \delta^e p^e_t E_t - \widetilde{C}_{t+1}] && (86)\\ & - (\pi p\widetilde{Q}_{t+1} - rB_t - \delta p K_t - \delta^e p^e_t E_t) && (87)\\ =& I_{t+1} + C_{t+1} - pQ_{t+1} + rB_t - rD_t = r(B_t - D_t), && (88) \end{aligned}$$

so that, as before, profits for the bank accrued from the interest differential between loans and deposits.

3.2 Approximate continuous-time Markov chain dynamics

The second ingredient of the approach consists in approximating the discrete-time transitions between types by a two-dimensional continuous-time Markov chain with state (N^1_t, M^1_t), that is, the numbers of aggressive firms and non-investor households at time t, and state space $\{0, 1, \ldots, N\} \times \{0, 1, \ldots, M\}$. Accordingly, we assume that the Markov chain at state (n, m) can jump to one of four neighbouring states $(n \pm 1, m \pm 1)$ with transition rates given by

$$\begin{aligned} d^f(n) &= \mu^f n, & b^f(n) &= \lambda^f(N-n) \\ d^h(m) &= \mu^h m, & b^h(m) &= \lambda^h(M-m) \end{aligned} \tag{89}$$

In other words, a jump from n to $n-1$, corresponding to the "death" of a type 1 firm, occurs in an small time interval dt with probability $d^f(n)dt$ obtained as the probability of an individual firm to transition from type 1 to type 2, which is given by μ^f according to (27), multiplied by the number n of firms currently of type 1. Similarly, a jump from n to $n+1$, corresponding to the "birth" of a type 1 firm, occurs in an small time interval dt with probability $b^f(n)dt$ obtained as the probability λ^f of an individual firm to transition from type 2 to type 1 multiplied by the number $(N-n)$ of firms currently of type 2. The death and birth transition rates for households are obtained analogously. Observe that these calculations for transition rates assume that the change in type for different firms and households are independent random events, thus the multiplication of each individual transition probability by the number of agents undergoing that transition.

The third and final ingredient consists in approximating the solution of the master equation (ME), namely the equation governing the time evolution of the probability

$$P(n, m; t) = \text{Prob}\left(N^1_t = n, M^1_t = m\right). \tag{90}$$

As shown in the Appendix, assuming that the numbers of firms and households of type 1 at time t can be written as

$$N^1_t = N\phi^f(t) + \sqrt{N}\xi^f(t), \qquad M^1_t = M\phi^h(t) + \sqrt{M}\xi^h(t) \tag{91}$$

for determinist functions $\phi^f(t)$ and $\phi^f(t)$ corresponding to their trends and stochastic processes $\xi^f(t)$ and $\xi^h(t)$ for random fluctuations around the trend, we obtain the following ordinary differential equations

$$\frac{d\phi^f}{dt} = \lambda^f - (\lambda^f + \mu^f)\phi^f, \qquad \frac{d\phi^h}{dt} = \lambda^h - (\lambda^h + \mu^h)\phi^h, \tag{92}$$

from which it is easy to see that

$$\phi^f(t) = \frac{\lambda^f}{\lambda^f + \mu^f} + e^{-(\lambda^f+\mu^f)t}\left(\phi^f(0) - \frac{\lambda^f}{\lambda^f + \mu^f}\right) \Rightarrow \phi^f_\infty := \lim_{t\to\infty} \phi^f(t) = \frac{\lambda^f}{\lambda^f + \mu^f} \tag{93}$$

$$\phi^h(t) = \frac{\lambda^h}{\lambda^h + \mu^h} + e^{-(\lambda^h+\mu^h)t}\left(\phi^h(0) - \frac{\lambda^h}{\lambda^h + \mu^h}\right) \Rightarrow \phi^h_\infty := \lim_{t\to\infty} \phi^h(t) = \frac{\lambda^h}{\lambda^h + \mu^h}. \tag{94}$$

Moreover, the probability densities of the random fluctuations satisfies two associated Fokker-Planck equations of the form (119), from which it follows that the fluctuations are asymptotically Gaussian distributed with means equal to zero and variances given by

$$\sigma_f^2 = \frac{\mu^f\lambda^f}{(\lambda^f + \mu^f)^2}, \qquad \sigma_h^2 = \frac{\mu^h\lambda^h}{(\lambda^h + \mu^h)^2}. \tag{95}$$

The fractions $n_t^1 = N_t^1/N$ and $m_t^1 = M_t^1/M$ can therefore be approximated by stochastic differential equations of the form

$$dn_t^1 = (\lambda^f + \mu^f)\left(\frac{\lambda^f}{\lambda^f + \mu^f)} - n_t^1\right)dt + \sqrt{\frac{2\mu^f\lambda^f}{N(\lambda^f + \mu^f)}}dW_t^f, \tag{96}$$

$$dm_t^1 = (\lambda^f + \mu^f)\left(\frac{\lambda^h}{\lambda^h + \mu^h)} - m_t^1\right)dt + \sqrt{\frac{2\mu^h\lambda^h}{N(\lambda^h + \mu^h)}}dW_t^h, \tag{97}$$

for independent Brownian motions (W_t^f, W_t^h).

Summing up, the mean-field model consists of the deterministic evolutions (55), (59)-(60), (76)-(77) for the 8 state variables $(\overline{k}^z, \overline{b}^z, \overline{e}^z, \overline{d}^z)$, with $z = 1, 2$, coupled with the stochastic evolution (96)-(97) for the fractions of firms and households of type 1 (with the corresponding rebalancing after each change in type according to expressions (50) and (51)). In other words, the mean-field model corresponds to a 10-dimensional random dynamical system. By comparison, the full agent-based model requires the calculation of four state variables $(k_t^n, b_t^n, e_t^n, z_t^n)$ for each firm and three state variables (d_t^m, e_t^m, z_t^m) for each household.

4 Numerical Simulations

In this section we illustrate the properties of the model by simulating both the full agent-based model and the mean-field approximation using the base parameters described in Table 2. The parameters were chosen consistently with the assumption that the discrete-time equations in the model correspond to quarterly updates, that is, the basic time period in the model is 0.25 years. In particular, the one-period depreciation rate δ, the dividend yield δ^e, and the interest rate r were chosen consistently with annualized rates of 4% for each variable. For the agent-based simulations, we take $p \equiv 1.4$, $p_0^e = 1$ and initialize the aggregate balance sheet items for firms at $pK_0 = 1400$, $B_0 = 667$, $E_0 = 333$, leading to initial aggregate net worth of the firm sector equal to $V_0^F = 400$, and aggregate balance sheet items the household sector at $D_0 = 1067$ and $p_0^e E_0 = 333$, leading to initial aggregate net worth of the household sector equal to $V_0^H = 1400$ [10]. We then assume these aggregate amounts are uniformly distributed among individual firms and households respectively.

Symbol	Value	Description
N	1000	number of firms
M	4000	number of households
a	1	labour productivity
c	1	unit labour cost
χ	1.4	markup factor
α_1	0.575	profit elasticity of investment for aggressive firms
α_2	0.4	profit elasticity of investment for conservative firms
β	0.16	utilization elasticity of investment
γ	0.05	debt elasticity of investment
r	0.01	one-period interest rate on loans and deposits
δ	0.01	one-period depreciation rate
δ^e	0.01	one-period dividend yield
s_1^y	0.15	propensity to save from income for non-investors
s_2^y	0.4	propensity to save from of income for investors
s_1^v	0.85	propensity to save from wealth for non-investors
s_2^v	0.85	propensity to save out of wealth for investors
μ^f	0.6	transition probability from aggressive to conservative type for firms
λ^f	0.4	transition probability from conservative to aggressive type for firms
μ^h	0.2	transition probability from non-investors to investors type for households
λ^h	0.3	transition probability from investors to non-investor type for households
ϖ	0.6	proportion of external financing for firms obtained issuing new debt
φ	0.5	proportion of investor household wealth allocated to stocks

Table 2: Baseline parameter values

We first compare the number of firms and households of each type obtained from the agent-based simulation and the mean-field approximation in Figure 1. Next in Figure 2 we compare the time evolution for equity prices and nominal output obtained from each method. We can observe a close match between the computationally intensive agent-based model and its mean-field approximation, both in terms of population fractions of each type of agent and in the resulting aggregate variables represented by the equity prices and output.

Next in Figures 3 to 8 we use the mean-field approximation to perform a series of sensitivity tests with respect to several discretionary parameters. Starting with Figure 3, we see that, as expected, the return on equity decreases linearly with the dividend yield δ^e. We also see that the growth rate of output increases with the dividend yield. This happens because, in our model, an increase in divided yield leads to higher disposable income of households and consequently higher consumption, whereas the offsetting decrease in

[10] We also assume a constant level of cash reserves for the bank $R_0 = 400$, so that the initial net worth of the bank implied by the aggregate balance sheets of firms and households is $V_0^B = 0$.

aggregate investment is less pronounce, as firms can borrow the necessary amount to finance investment. The base value $\delta^e = 0.01$, corresponding to an annual dividend yield of 4%, is compatible with average observed yields and leads to an average 2.7% growth rate in equity and average 3.0% growth rate of nominal output in our model.

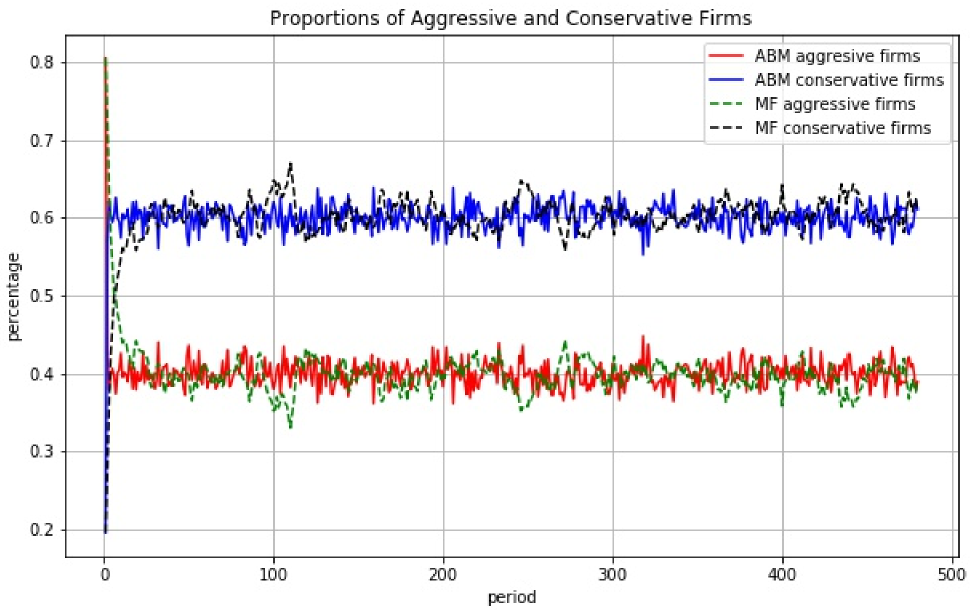
The sensitivity test for the proportion ϖ of external financing that firms raised through new debt is shown in Figure 4 and confirms that the base value chosen for this parameters lie in a range where aggregate variables such as equity prices and output are not only realistic but relatively stable with respect to small changes in the parameters. The results in Figure 4 suggest that the value of a firm, here reflected by the equilibrium equity price, is independent from the particular mix of debt and equity used to finance its operations. That this seems to break down for $\varpi < 0.4$ is puzzling and merits further investigation. Figure 5 shows a similar result for the parameter φ, where it is interesting to see that the volatility of equity prices tends to increase both when investors put all their wealth in stocks (namely $\varphi \to 1$) or none of their wealth in stocks (namely $\varphi \to 0$).

Figure 6 shows the sensitivity tests for the profit elasticity parameter, where for the purposes of the test we took $\alpha = \alpha_1 = \alpha_2$, that is to say, equal for all firms. As we can observe in the middle panel, the growth rate of output increases with α, since a higher value for this parameter leads to higher investment by firms. On the other hand, as the top panel illustrates, increasing value for α have a negative effect on the growth rate of equity prices, as firms need to raise more funds for external financing and therefore increase the supply of equities. For the same reason, higher values of α lead to higher debt-to-output ratios, as firms also need to borrow more to finance investment. Our base parameters reflect a choice where the level responsiveness of investment to past profits is high enough to promote growth but not as high as to compromise the financial viability of firms.

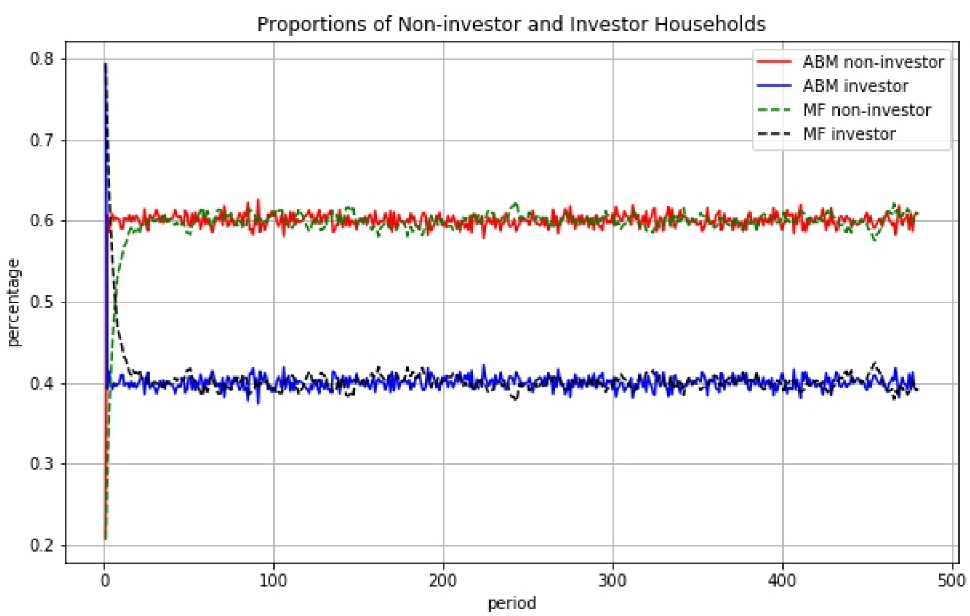
Figure 7 shows similar results for the saving rate from income s^y, which we assumed to be the same for all households for the purpose of the sensitivity tests. As expected, we see in the bottom panel that aggregate output decreases with savings from income, as this shifts household spending from consumption to accumulation of bank deposits and stocks, thereby raising the growth rate of equity as shown in the top panel of the same figure. This effect is all the more pronounced when we consider the saving rate from wealth s^v in Figure 8. As we can see, a high propensity to spend accumulated wealth (correspondingly low s^v) leads to high growth rate but disastrous equity prices (both volatile and with negative returns). Conversely, total reinvestment of wealth (namely $s^v \to 1$) leads to high returns (but also high volatility) in stock prices, but precipitously low growth for the economy as a whole. Our base line parameters reflect a compromise between these two conflicting tendencies.

5 Conclusion and further work

We have proposed a mean-field approximation to a stock-flow consistent agent-based model with heterogeneous firms and households. The approximation is inspired by the earlier work in Di Guilmi et al. (2010), Delli Gatti et al. (2012), but differs from these papers in two fundamental aspects. First, we take the transition rates between types to be exogenous and constant, as this is the case for which the solution method for the master equation described in the Appendix applies. Secondly, we introduce an addition rebalancing of mean-field variables (namely equations (50) and (51)) that is imposed by stock-flow consistency and seems to have been previously neglected.

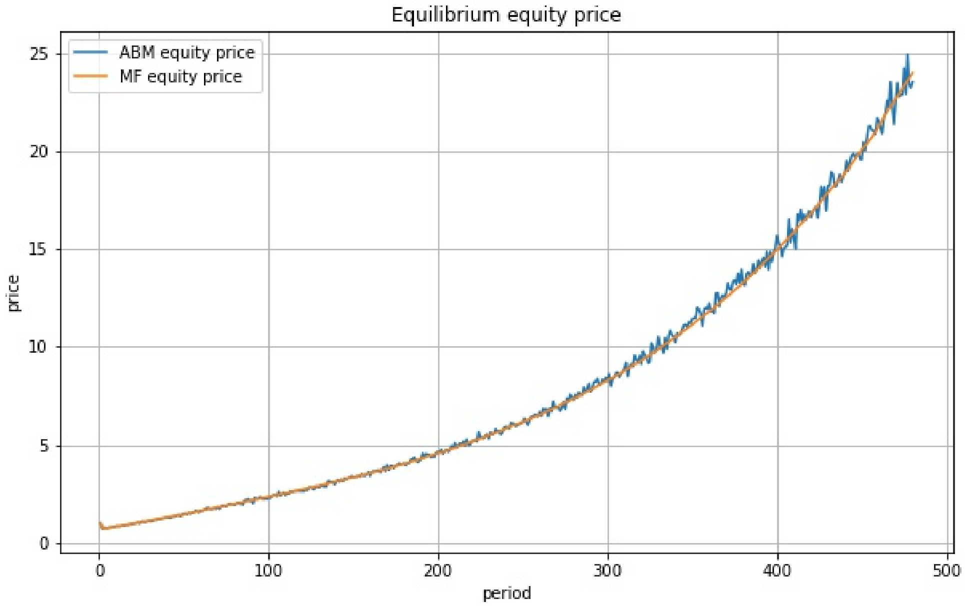


(a) The average fraction of type 1 (aggressive) firms is 0.4

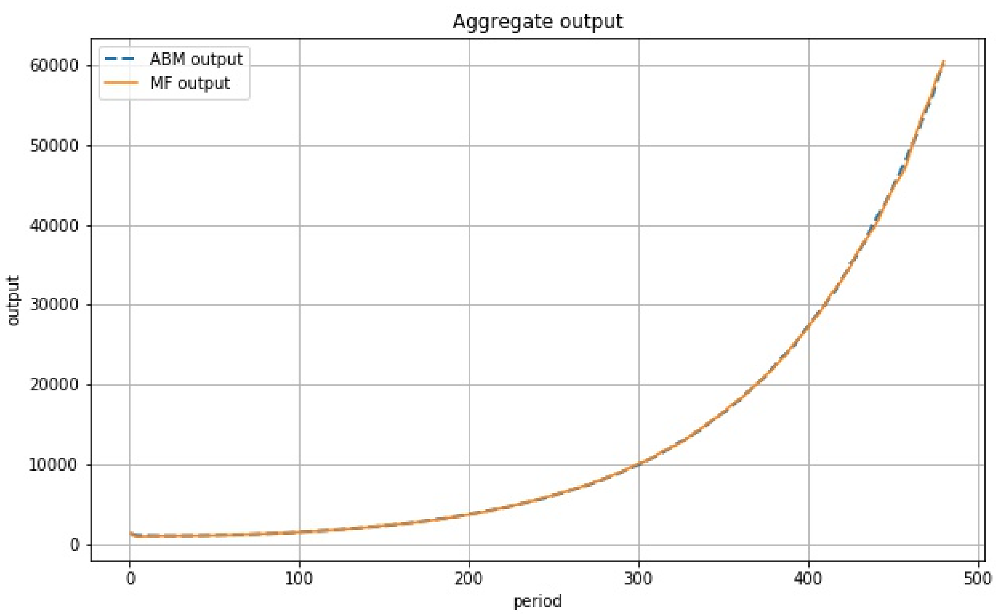


(b) The average fraction of type 1 (non-investor) households is 0.6.

Figure 1: Number of firms and households of each type obtained through agent-based simulations (ABM) and mean-field approximations (MF).

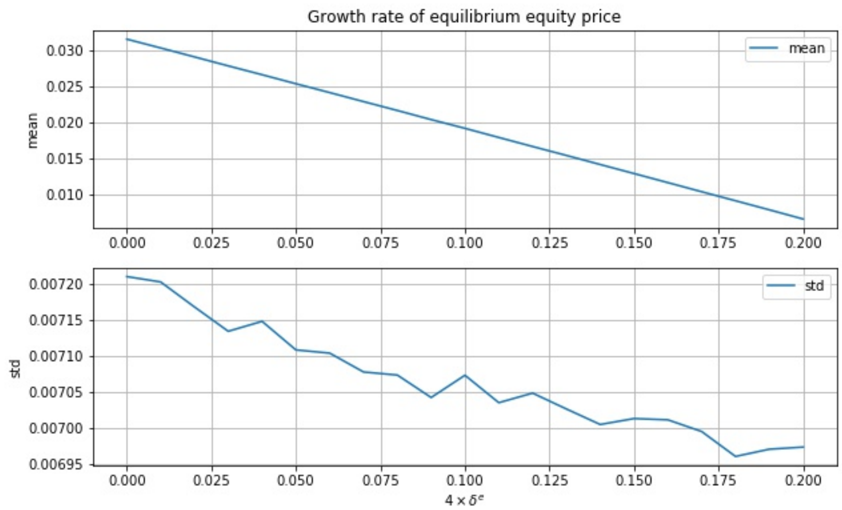


(a) The initial equity price is $p_0^e = 1$, the average annual return over the 120 years period is 2.7% for both the ABM simulation and MF approximation.

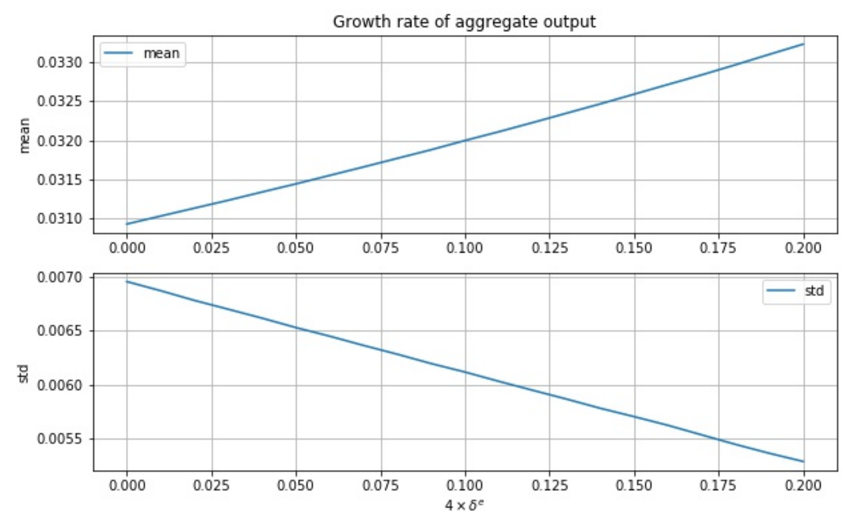


(b) The initial output is $Q_0 = 1000$, the average annual growth rate over the 120 years period is 2.9% for both the ABM simulation and the MF approximation.

Figure 2: Comparison between aggregate variables in the agent-based model (ABM) and mean-field approximation (MF).

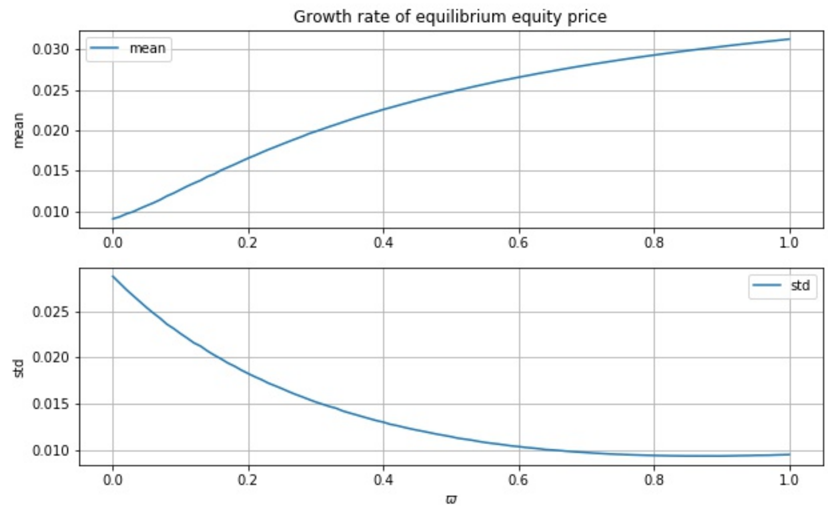


(a) Growth rate and standard deviation of equity price.

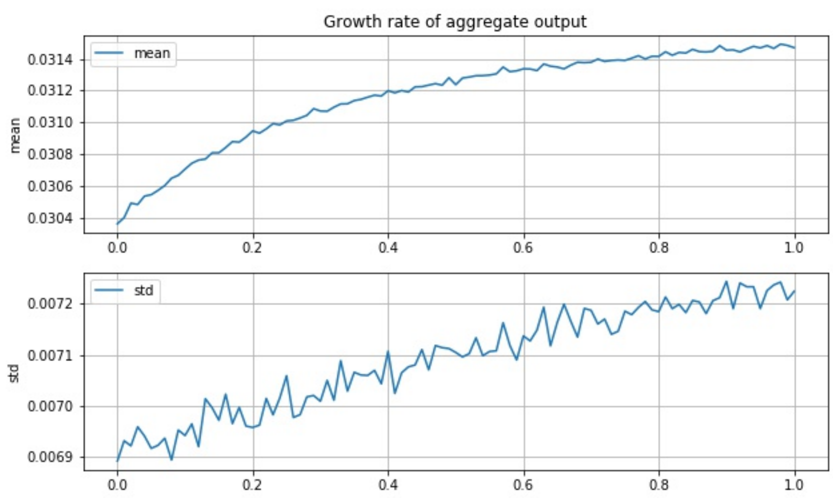


(b) Growth rate and standard deviation of aggregate output.

Figure 3: Sensitivity of equity price and aggregate output to the dividend yield δ^e. Recall that the annualized dividend yield is given by $4 \times \delta^e$.

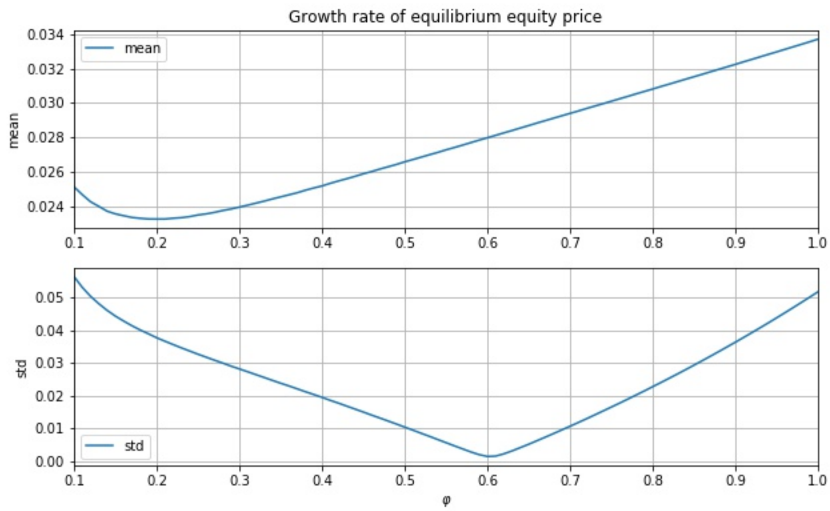


(a) Growth rate and standard deviation of equity price.

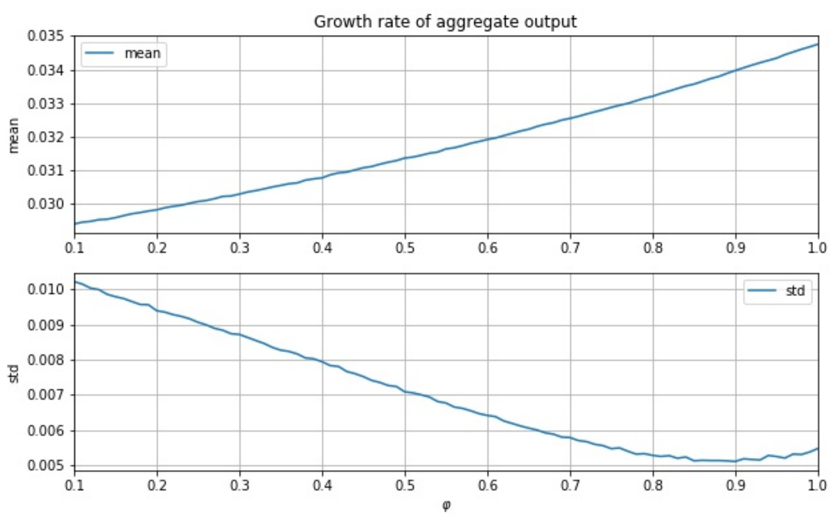


(b) Growth rate and standard deviation of aggregate output.

Figure 4: Sensitivity of equity price and aggregate output to the proportion ϖ of external financing raised by debt.

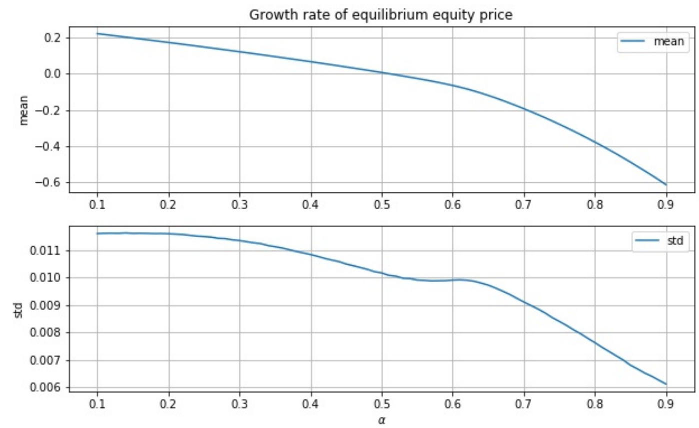


(a) Growth rate and standard deviation of equity price.

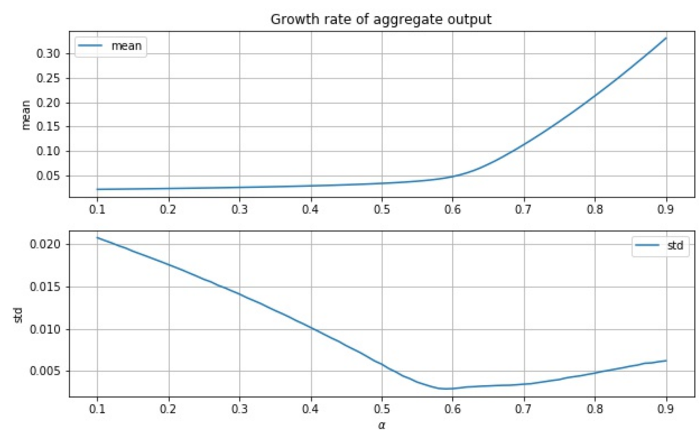


(b) Growth rate and standard deviation of aggregate output.

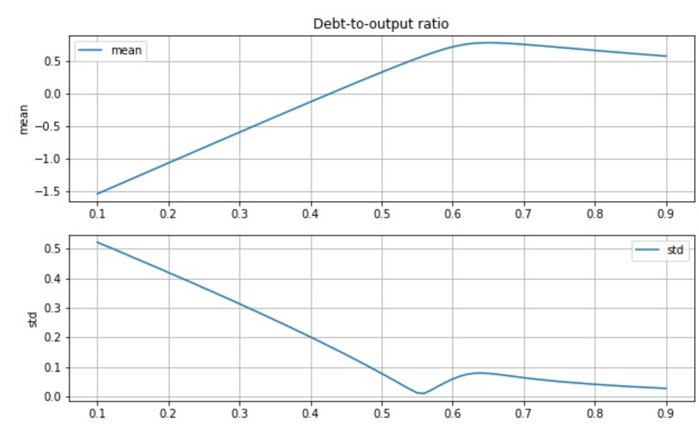
Figure 5: Sensitivity of equity price and aggregate output to the proportion φ of household wealth invested in stocks.



(a) Growth rate and standard deviation of equity price.

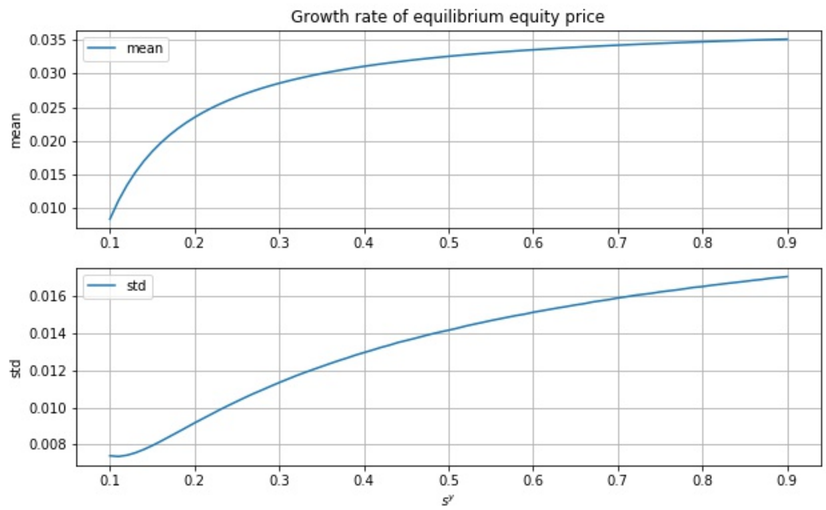


(b) Growth rate and standard deviation of aggregate output.

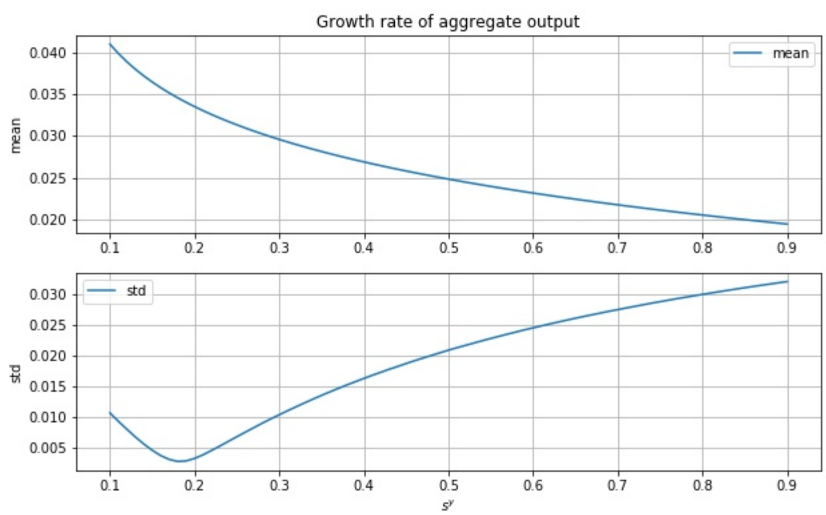


(c) Debt-to-output ratio

Figure 6: Sensitivity of equity price, aggregate output, and debt-to-output ratio to the profit elasticity of investment α for firms.

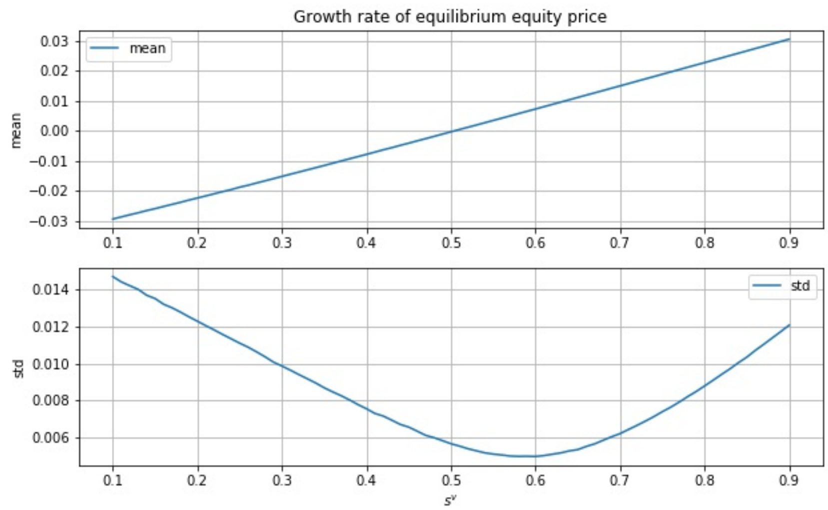


(a) Growth rate and standard deviation of equity price.

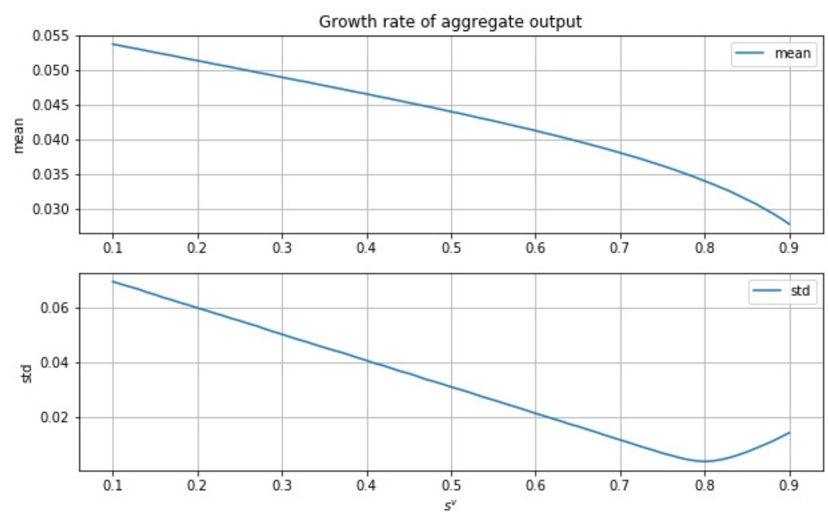


(b) Growth rate and standard deviation of aggregate output.

Figure 7: Sensitivity of equity price and aggregate output to the savings rate from income s^y for households.



(a) Growth rate and standard deviation of equity price.



(b) Growth rate and standard deviation of aggregate output.

Figure 8: Sensitivity of equity price and aggregate output to the savings rate from wealth s^v for households.

Our model for different firms is motivated by Di Guilmi and Carvalho (2017), except that we classify firms into aggressive and conservative, rather than self-financing and non-self-financing. In other words, the amount a firm decide to invest depends on an inherent property (for example the "animal spirits" of its managers), rather than its financial position, which is then determined afterwards depending on the overall state of the economy. Similarly, our model for different households is motivated Carvalho and Di Guilmi (2014), except that we classify households into non-investors and investors, rather than borrowing and non-borrowing. In other words, a household's decision to invest on the stock market depends on an inherent property (for example the degree of risk aversion), rather than its financial position.

With these two modifications, we obtain remarkable accuracy in the MF approximation of aggregate variables when compared with the simulations of the underlying ABM. We then use the MF approximation to perform a series of sensitivity tests for the model with respect to some of its parameters, notably the dividend rate δ^e, the proportion ϖ of external financing that firms raise from new debt, the proportion φ of household wealth invested in the stock market, the elasticity α of invest to profits and the propensity s^y for households to save from income. These tests allow us to investigate the range of parameters that result in plausible behaviour for the aggregate variables in the model.

For example, a sufficiently high value for the fraction ϖ of external finance raised by issuing debt or the fraction φ of household wealth invested in equities leads to stable equity prices, characterized by high return and low volatility as shown in Figure 2. Accordingly, we can simulate models with more turbulent stock markets, that is to say characterized by crashes and periods of high volatility, by lowering the values of these parameters. A natural follow up question, motivated by Minskys Financial Instability Hypothesis (see Minsky (1982)), is whether a suitable extension of the model can allow for a stable scenario to evolve into an unstable one.

One way to achieve this is to introduce more interactions between the agents than we considered in this paper. Specifically, we can let the death and birth probabilities for firms in Section 3.2 to be of the form

$$d^f(n) = \mu^f \eta^1\left(\frac{n}{N}\right) n, \quad b^f(n) = \lambda^f \eta^2\left(\frac{n}{N}\right)(N-n), \tag{98}$$

for functions $\eta^1(\cdot)$ and $\eta^2(\cdot)$ related to the relative gains from being of one type versus another, and the solution method presented in Aoki (2002) still applies to this type of transition probabilities. For example, the functions $\eta^1(\cdot)$ and $\eta^2(\cdot)$ can be related to profits for firms of different types, so that higher profits for aggressive firms would lead to a more firms becoming aggressive, and it is plausible to conjecture that this kind of endogenous transition probabilities can generate instability from periods of stability, but we defer this investigation to future work.

Acknowledgements: The authors thank Corrado Di Guilmi, Marco Pangallo and John Muellbauer for comments and discussions, as well as the participants of the Econophysics Colloquium (São Paulo, July 2016), the Research in Options Conference (Rio de Janeiro, December 2016), the Bachelier Colloquium 2017 (Metabief, January 2017), the INET Researcher Seminar (Oxford, March 2017) and the University of Southern California Mathematical Finance Colloquium (Los Angeles, April 2017), were portions of this work were presented.

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A Approximate Solution to the Master Equation

We adjust the solution method used in Di Guilmi et al. (2010), which is itself adapted from Aoki (2002) and earlier references, to the case where there are two types of firms and two types of households. Let $(N_t^1, M_t^1) \in \{0, 1, \dots, N\} \times \{0, 1, \dots, M\}$ denote the number of firms of type 1 and the number of households of type 1, respectively. It follows from the Markov property that the joint probability

$$P(n, m; t) = \text{Prob}\left(N_t^1 = n, M_t^1 = m\right), \tag{99}$$

satisfies the so-called master equation, that is,

$$\begin{aligned}\frac{\partial P(n, m; t)}{\partial t} = {} & d^f(n+1)P(n+1, m; t) + b^f(n-1)P(n-1, m; t) \\ & + d^h(m+1)P(n, m+1; t) + b^h(m-1)P(n, m-1; t) \\ & - [d^f(n) + b^f(n) + d^h(m) + b^h(m)]P(n, m; t),\end{aligned} \tag{100}$$

with the obvious modifications at the boundaries $n = m = 0$, $n = N$, and $m = M$. Here the "death" and "birth" transition probabilities are defined in (89). Assuming that firms and households choose their type independently from each other, we have that

$$P(n, m; t) = P(n, t)P(m, t), \tag{101}$$

where $P(n, t) = \text{Prob}\left(N_t^1 = n\right)$ and $P(m, t) = \text{Prob}\left(M_t^1 = m\right)$. Substituting (101) on both sides of (100) leads to

$$\begin{aligned}\frac{\partial P(n, t)}{\partial t}P(m, t) + P(n, t)\frac{\partial P(m, t)}{\partial t} = {} & \left(d^f(n+1)P(n+1, t) + b^f(n-1)P(n-1, t)\right)P(m, t) \\ & + \left(d^h(m+1)P(m+1, t) + b^h(m-1)P(m-1, t)\right)P(n, t) \\ & - [d^f(n) + b^f(n)]P(n, t)P(m, t) - [d^h(m) + b^h(m)]P(n, t)P(m, t).\end{aligned} \tag{102}$$

Assuming further that $P(n, t) \neq 0$ and $P(m, t) \neq 0$ for all n, m, we find that (102) decouples into the following equations:

$$\begin{aligned}\frac{\partial P(n, t)}{\partial t} = {} & d^f(n+1)P(n+1, t) + b^f(n-1)P(n-1, t) \\ & - [d^f(n) + b^f(n)]P(n, t),\end{aligned} \tag{103}$$

$$\begin{aligned}\frac{\partial P(m, t)}{\partial t} = {} & d^h(m)P(m+1, t) + b^h(m-1)P(m-1, t) \\ & - [d^h(m) + b^h(m)]P(m, t),\end{aligned} \tag{104}$$

which are identical to the master equation analyzed in Di Guilmi et al. (2010). We proceed the analysis in terms of firms, with the results for households following from obvious modifications. As in Di Guilmi et al. (2010), for a generic function $a(n)$ define the lead and lag operators as

$$L[a(n)] = a(n+1), \qquad L^{-1}[a(n)] = a(n-1), \tag{105}$$

so that we can rewrite (103) as

$$\frac{\partial P(n, t)}{\partial t} = (L - 1)[d^f(n)P(n, t)] + (L^{-1} - 1)[b(n)P(n, t)]. \tag{106}$$

Applying Taylor expansions to $a(n+1)$ and $a(n-1)$ at n we find that the operators $(L-1)$ and $(L^{-1}-1)$ can be written as:

$$\begin{aligned}(L-1)[a(n)] = a(n+1) - a(n) &= [a(n) + a'(n) + \frac{a''(n)}{2} + \ldots] - a(n) \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k a(n)}{dn^k}\end{aligned} \tag{107}$$

and

$$\begin{aligned}(L^{-1}-1)[a(n)] = a(n-1) - a(n) &= [a(n) - a'(n) + \frac{a''(n)}{2} + \ldots] - a(n) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{d^k a(n)}{dn^k}\end{aligned} \tag{108}$$

Using the ansatz (91), we will now rewrite (106) in terms of $\phi(t) := \phi^f(t)$ and $\xi(t) = \xi^f(t)$. Observe first that, since $\phi(t)$ is assumed to be deterministic, we can write

$$P(n,t) = Q(\xi,t) = Q(\xi(t),t), \tag{109}$$

where $Q(\xi,t)$ is the distribution of the stochastic process $\xi(t)$. This leads to

$$\begin{aligned}\frac{\partial P(n,t)}{\partial t} &= \frac{\partial Q(\xi,t)}{\partial t} + \frac{\partial Q(\xi,t)}{\partial \xi}\frac{d\xi}{dt} \\ &= \frac{\partial Q(\xi,t)}{\partial t} - \sqrt{N}\frac{\partial Q(\xi,t)}{\partial \xi}\frac{d\phi}{dt},\end{aligned} \tag{110}$$

where we differentiated the relation

$$n = N\phi(t) + \sqrt{N}\xi$$

with respect to t at constant n to obtain

$$\frac{d\xi}{dt} = -\sqrt{N}\frac{d\phi}{dt}.$$

Next observe that the transition probabilities can be expressed as

$$d(n) = d(\xi,t) = \mu(N\phi(t) + \sqrt{N}\xi) \tag{111}$$

$$b(n) = b(\xi,t) = \lambda(N - N\phi(t) - \sqrt{N}\xi) \tag{112}$$

where $\mu := \mu^f$ and $\lambda := \lambda^f$. Finally, since $a(n) = a(\xi,t) = a(N\phi(t) + \sqrt{N}\xi)$ we have that

$$\frac{da(n)}{dn} = \frac{1}{\sqrt{N}}\frac{da(\xi)}{d\xi}, \tag{113}$$

so that (107) and (108) become

$$(L-1)[a(\xi)] = \sum_{k=1}^{\infty} \frac{1}{k! N^{\frac{k}{2}}} \frac{d^k a(\xi)}{d\xi^k} \tag{114}$$

$$(L^{-1}-1)[a(\xi)] = \sum_{k=1}^{\infty} \frac{(-1)^k}{k! N^{\frac{k}{2}}} \frac{d^k a(\xi)}{d\xi^k} \tag{115}$$

Inserting (110) in the left-hand side of (106) and (111)-(115) in the right-hand side we obtain

$$\begin{aligned}\frac{\partial Q}{\partial t} - \sqrt{N}\frac{\partial Q}{\partial \xi}\frac{d\phi}{dt} &= (L-1)[d(\xi,t)Q(\xi,t)] + (L^{-1}-1)[b(\xi,t)Q(\xi,t)] \\ &= (L-1)[\mu(N\phi(t)+\sqrt{N}\xi)Q(\xi,t)] \\ &\quad + (L^{-1}-1)[\lambda(N - N\phi(t) - \sqrt{N}\xi)\cdot Q(\xi,t)] \\ &= \left(\sum_{k=1}^{\infty}\frac{1}{k!N^{\frac{k}{2}}}\frac{d^k}{d\xi^k}\right)[\mu(N\phi(t)+\sqrt{N}\xi)Q(\xi,t)] \\ &\quad + \left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!N^{\frac{k}{2}}}\frac{d^k}{d\xi^k}\right)[\lambda(N - N\phi(t) - \sqrt{N}\xi)\cdot Q(\xi,t)]\end{aligned} \tag{116}$$

Collecting terms of order $\sqrt{N}$ in the equation above leads to[11]

$$\frac{d\phi}{dt} = \lambda - (\lambda+\mu)\phi, \tag{117}$$

whose solution is readily found to be[12]

$$\phi(t) = \frac{\lambda}{\lambda+\mu} + e^{-(\lambda+\mu)t}\left(\phi(0) - \frac{\lambda}{\lambda+\mu}\right). \tag{118}$$

Similarly, collecting terms of order 1 in (116) leads to

$$\frac{\partial Q}{\partial t} = (\mu+\lambda)\frac{\partial(\xi Q)}{\partial \xi} + \frac{\mu\phi + \lambda(1-\phi)}{2}\frac{\partial^2 Q}{\partial \xi^2}. \tag{119}$$

We therefore see that ξ admits an asymptotically stationary distribution

$$Q_\infty(\xi) := \lim_{t\to\infty} Q(\xi,t) \tag{120}$$

satisfying

$$\frac{\partial^2 Q}{\partial \xi^2} = -\frac{2(\mu+\lambda)}{\mu\phi_\infty + \lambda(1-\phi_\infty)}\frac{\partial(\xi Q)}{\partial \xi}, \tag{121}$$

where

$$\phi_\infty = \lim_{t\to\infty}\phi(t) = \frac{\lambda}{\lambda+\mu}. \tag{122}$$

[11]At this point in the derivation, the authors in Di Guilmi et al. (2010) inexplicably change the transition rate (111)-(112) to the form given in their equation (13A.19), which coincide with the transition rates for a *different* model described on page 23 of Aoki (2002). The analogue of equations (117) and (119) thus obtained Di Guilmi et al. (2010) coincides with the corresponding equations on page 37 of Aoki (2002), but are *not* related to the model described in Di Guilmi et al. (2010) up to this point.

[12]Up to here the derivation also works for time-dependent transition rates $\lambda(t)$ and $\mu(t)$. However, the solution (118), and the corresponding asymptotic value $\phi_\infty = \lambda/(\lambda+\mu)$, *only* hold for constants λ and μ. The same is true for (13.30) in Di Guilmi et al. (2010), which *only* holds as a solution to their (13.27) in case λ and γ (their analogue of μ) are constant, so it is unclear how the authors obtain steady-state results that depend on the ϕ_∞ (such as the output dynamics in their (13.32)) when the transition rates are time-dependent as implied by their equations (13.16) and (13.17). This is even more problematic for state-dependent transition rates, as suggested in equations (13.33)-(13.34) in Di Guilmi et al. (2010), since in this case λ and μ would be functions of ξ in (116) and would *not* lead to equation (117) for ϕ.

Integrating (121) we find that[13]

$$Q_\infty(\xi) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\xi^2}{2\sigma^2}}, \tag{123}$$

where

$$\sigma^2 = \frac{\mu\lambda}{(\mu+\lambda)^2}. \tag{124}$$

[13] The same remark about transition rates applies here: equation (119) holds for time-dependent rates $\lambda(t)$ and $\mu(t)$ (but *not* for state-dependent ones), whereas the stationary solution (121) *only* holds for constants λ and μ.