Derivation of the Price of Bond in the Recovery of Market Value Model

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1 Recovery models

For the analysis of reduced-form pricing models, there are some recovery assumptions, such as the zero recovery, recovery of treasury, recovery of par and the recovery of market value. Consider the doubly stochastic model with a stochastic intensity λ_t and assume that the recovery rate R_t at time t. Assume that the default time $\tau > 0$, the price at time zero of a defaultable bond with maturity time T is the sum of a recovery term and a survival term. It is given by

$$\begin{aligned}
\bar{P}_{0T} &= E^Q[e^{-\int_0^\tau r_s ds} R_\tau 1_{\tau \leq T} + e^{-\int_0^T r_s ds} 1_{\tau > T}] \\
&= E^Q[E^Q[e^{-\int_0^\tau r_s ds} R_\tau 1_{\tau \leq T} | F_T]] + E^Q[e^{-\int_0^T r_s ds} 1_{\tau > T}] \\
&= E^Q[\int_0^T e^{-\int_0^\tau (r_s + \lambda_s ds} R_u \lambda_u du] + E^Q[e^{-\int_0^T r_s + \lambda_s ds}] \\
&= \int_0^T E^Q[R_u \lambda_u e^{-\int_0^u r_s + \lambda_s ds}] du + E^Q[e^{-\int_0^T r_s + \lambda_s ds}]
\end{aligned} \tag{1}$$

- **Zero Recovery**
  In the zero recovery model, R_t=0 for all t $\geq$ 0, so the expression of $\bar{P}_{0T}$ is the survival term of Equation (1).

$$\bar{P}_{0T} = E^Q[e^{-\int_0^\tau r_s + \lambda_s ds}] \tag{2}$$

- **Recovery of treasury**
  In the recovery of treasury, the recovery rate is in the form $R_t = RP_{0T}$, where $0 < R \leq 1$. Then

$$\begin{aligned}
\bar{P}_{0T}^{RT} &= R \int_0^T E^Q[P_{uT} \lambda_u e^{-\int_0^u (r_s + \lambda_s) ds}] du + \bar{P}_{0T} \\
&= RE^Q[e^{-\int_0^T r_s ds} \int_0^T \lambda_u e^{-\int_0^u \lambda_s ds} du] + \bar{P}_{0T} \\
&= RE^Q[e^{-\int_0^u r_s ds}(1 - e^{-\int_0^T \lambda_s ds})] + \bar{P}_{0T} \\
&= RP_{0T} + (1 - R)\bar{P}_{0T}
\end{aligned} \tag{3}$$

- **Recovery of par**
  In the recovery of par model, R_t = R, where $0 < R < 1$, for all t $\geq$ 0.

$$\bar{P}_{0T}^{RT} = R \int_0^T E^Q[\lambda_u e^{-\int_0^u (r_s + \lambda_s) ds}] du + \bar{P}_{0T} \tag{4}$$

For the recovery of market value, assume that, at each time t, conditional on all information available up to but not including time t, a specified risk-neutral mean fraction L_t of market value is lost if default occurs at time t. One can consider the (1-L_t) as the recovery rate R_t at time t. The formula of the price at time t of a defaultable bond is derived in the following section.

2 Derivation of the Price in RMV Model

2.1 Discrete-time Valuation

Under risk-neutral probability measure, let h_t be the hazard rate for default at time t and let L_t be the expected fractional loss in market value if default were to occur at time t, conditional on the information available up to time t (Duffie and Singleton, 1999). Suppose that, X is paid at time T in the event of no default. Then, the initial market value of the defaultable claim to X is

$$P_{0T} = E_0^Q[e^{-\int_0^T R_t dt} X] \tag{5}$$

where E_0^Q denotes risk-neutral, conditional expectation at data 0.
Consider a defaultable case that, $X_{t+\tau}$ is paid at maturity date t+τ, and nothing before date t+τ. Let φ_t be the recovery in units of account in the event of default at time t, and r_t be the default-free short rate. The hazard rate h_t is also the conditional probability at time t of default between t and t+1 given the information available at time t in the event of no default by t.
Therefore, if the asset has not defaulted by time t, its market value P_t is the present value of receiving φ_{t+1} in the event of default between t and t+1 plus the present value of receiving V_{t+1} in the event of no default, that is

$$P_t = h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(P_{t+1}) \tag{6}$$

where $E_t^Q(.)$ denotes the expectation under Q, conditional on information available to investors at date t (Duffie and Singleton, 1999).
By recursively solving Equation () forward over the life of the bond, P_t can be expressed equivalently as

$$\begin{aligned} P_t &= h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(P_{t+1}) \\ &= E_t^Q[\sum_{j=0}^{\tau-1} h_{t+j} e^{-\sum_{k=0}^{j} r_{t+k}} \varphi_{t+j+1} \prod_{l=0}^{j} (1 - h_{t+l-1})] + E_t^Q[e^{-\sum_{k=0}^{\tau-1} r_{t+k}} X_{t+\tau} \prod_{j=1}^{\tau} (1 - h_{t+j-1})] \end{aligned} \tag{7}$$

The equation can be simplified by taking the risk-neutral expected recovery at time s, where s¡t, in the event of default at time s+1, to be a fraction of the risk-nuetral expected survival-contingent market value at time s+1. Let R_s be the recovery rate at time s. Thus, RMV(Recovery of Market Value):

$$E_s^Q(\varphi_{s+1}) = R_s E_s^Q(P_{s+1}) \tag{8}$$

Substituting RMV into Equation () then get

$$\begin{aligned} P_t &= h_t e^{-r_t} R_t E_t^Q(P_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(P_{t+1}) \\ &= E_t^Q(P_{t+1})[h_t e^{-r_t} R_t + (1 - h_t) e^{-r_t}] \end{aligned}$$

By the approximation, we have $e^x \simeq 1 + x$, for small x. Then

$$\begin{aligned} e^{-Y_t} &= h_t e^{-r_t} R_t + (1 - h_t)e^{-r_t} \\ &= e^{-r_t}[h_t R_t + 1 - h_t] \\ &= e^{-r_t}\{h_t[R_t - 1] + 1\} \end{aligned}$$

$$\Rightarrow e^{-Y_t + r_t} = h_t(R_t - 1) + 1$$

$$\Rightarrow -Y_t + r_t = h_t(R_t - 1), \text{(By the approximation)}$$

$$\Rightarrow Y_t = r_t + h_t(1 - R_t)$$

Therefore, Equation (5) can be expressed as

$$\begin{aligned} P_t &= E_t^Q(P_{t+1})[h_t e^{-r_t} R_t + (1 - h_t)e^{-r_t}] \\ &= E_t^Q(P_{t+1})e^{-Y_t} \\ &= E_t^Q(P_{t+1})e^{-Y_t} \end{aligned}$$

Substituting Equation (8) and $Y_t = r_t + h_t(1 - R_t)$ into Equation (), then

$$\begin{aligned} P_t &= E_t^Q(e^{-\sum_{k=0}^{\tau-1} Y_{t+k}} X_{t+\tau}) \\ &= E_t^Q(e^{-\sum_{k=0}^{\tau-1} r_{t+k} + h_{t+k}(1 - R_{t+k}} X_{t+\tau}) \end{aligned}$$

For the inhomogeneous Poisson process reduced form model, $h_t = \lambda_t$. Therefore, for the discrete-time,

$$P_t = E_t^Q(e^{-\sum_{k=0}^{\tau-1} r_{t+k} + \lambda_{t+k}(1 - R_{t+k}} X_{t+\tau}) \tag{9}$$

2.2 Continuous-time Valuation

We fix a probability space (Ω, F, P) and a family $F_t : t \geq 0$ of $\sigma - algebras$ satisfying the usual conditions (Protter, 1990). A predictable short-rate process r is also fixed, so that it is possible at any time t to invest one unit of account in default-free deposits and "roll over" the proceeds until a later time s for a market value at that time of $e^{\int_t^s r_u du}$.
Suppose Z is paid at a stopping time τ, which is denoted by (Z,τ). In order to have the payment can be made based on currently available information, we assume that Z is F_t-measurable. Taking an equivalent martingale measure Q relative to the short-rate process r. Thus, the ex dividend price process U of any given (Z,τ) is defined by

$$U_t = \begin{cases} E_t^Q[e^{-\int_t^\tau r_u du} Z], & \text{if } t < \tau \\ 0, & \text{if } t \geq \tau \end{cases} \tag{10}$$

where E_t^Q denotes the expectation under the risk-neutral measure Q, given F_t (Duffie and Singleton, 1999).

Suppose (X,T) and (X',T') are two pairs of claims. The first claim (X,T) means that the issuer has to pay X at date T, and the second claim (X',T') defines the stopping time T' at which the issuer defaults and X' is paid to claimholders. Hence, the claim (Z,τ) can be re-written as

$$\tau = min(T, T'), \text{and} Z = X1_{\{T<T'\}} + X'1_{\{T\geq T'\}} \tag{11}$$

Suppose the default time T' has a risk-neutral default hazard rate process h, which means that the process N_t is

$$N_t = \begin{cases} 0, & \text{if } t < \tau \\ 1, & \text{if } t \geq \tau \end{cases} \tag{12}$$

If default occurs at time t, then we will suppose that the claim pays

$$X' = R_t U_{t-} \tag{13}$$

where $U_{t-} = \lim_{s\to t} U_s$ is the price of the lcaim "just before" default, and R_t is the random variable describing the recovery rate of market value of the claim at default (Duffie and Singleton, 1999).
Similarly to the discrete-time model, we guess the continuous-time formula is

$$\begin{aligned} P_t &= E_t^Q[e^{-\int_t^T Y_s ds} X] \\ &= E_t^Q[e^{-\int_t^T (r_s+\lambda_s R_s)ds} X] \end{aligned} \tag{14}$$

To confirm the formula above, we set a gain process G, after discounting at the short-rate process r, must be a martingale under Q.

$$G_t = e^{-\int_0^t r_s ds} P_t(1 - N_t) + \int_0^t e^{-\int_0^s r_u du} R_s P_{s-} dN_s \tag{15}$$

The first term is the survival term, and the second term is the discounted payout of the claim upon default. This Equation is similar to the Equation (6). P_T=X and G is a Q martingale.
Applying Ito's formula (Protter 1990) to Equation (15), using Equation (13), we can see that for G to be a Q martingale, it is necessary and sufficient that

$$P_t = \int_0^t (r_s + R_s\lambda_s)ds + m_t \tag{16}$$

for some Q martingale m. Since P jumps at most a countable number of times, we can replace V_{s-} in Equation (15) with P_s. Given the terminal boundary condition $P_T = X$, this implies Equation (14). (Duffie and Singleton, 1999)

3 Reference

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