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INDIFFERENCE PRICE WITH GENERAL SEMIMARTINGALES

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Using duality methods, we prove several key properties of the indifference price π for contingent claims. The underlying market model is very general and the mathematical formulation is based on a duality naturally induced by the problem. In particular, the indifference price π turns out to be a convex risk measure on the Orlicz space induced by the utility function.

KEY WORDS: indifference price, utility maximization, non locally bounded semimartingale, random endowment, incomplete market, Orlicz space, convex duality, convex risk measure.

1. INTRODUCTION

The main purpose of this paper is to study the indifference pricing framework in markets where the underlying traded assets are described by general semimartingales which are *not assumed to be locally bounded*. Following Hodges and Neuberger (1989), we define the *(seller) indifference price* $\pi(B)$ of a claim B as the implicit solution of the equation

$$(1.1)\quad \sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T H_t\,dS_t\right)\right] = \sup_{H\in\mathcal{H}^W} E\left[u\left(x+\pi(B)+\int_0^T H_t\,dS_t - B\right)\right],$$

where $x \in \mathbb{R}$ is the constant initial endowment, $T < \infty$ is a fixed time horizon while S is an $\mathbb{R}^d$-valued càdlàg semimartingale defined on a filtered stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ that satisfies the usual assumptions. The $\mathbb{R}^d$-valued portfolio process H belongs to an appropriate class $\mathcal{H}^W$ of admissible integrands defined in Section 2.1 through a random variable W that controls the losses incurred in trading. B is an $\mathcal{F}_T$-measurable random variable corresponding to a financial liability at time T and satisfies the integrability conditions discussed in Section 3.1.

Throughout the paper, the utility function u is assumed to be an increasing and concave function $u : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x\to-\infty} u(x) = -\infty$. Neither *strict monotonicity nor strict concavity are required*, but we exclude the case when u is constant on $\mathbb{R}$.

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In principle, a general way to compute the indifference price in (1.1) is to solve the two utility maximization problems, in the sense of finding the optimizers in the class of admissible integrands. Such optimizers then correspond to the optimal trading strategies that an investor should follow with or without the claim B, therefore providing a corresponding notion of *indifference hedging* for the claim. However, it is generally possible to employ duality arguments to obtain the optimal *values* for utility maximization problems under broader assumptions than those necessary to find their optimizer. Since these values are all that is necessary for calculating the indifference price itself, the main goal here is the pursuit of such duality results rather than a full analysis of the indifference hedging problem which is deferred to future work (even though some partial results in this direction are provided in Proposition 3.11).

The key to establish such duality results is to choose convenient dual spaces as the ambient for the domains of optimization. Our approach is to use the Orlicz space $L^{\widehat{u}}$—and its dual space $L^{\widehat{\Phi}}$—that arises naturally from the choice of the utility function u and was previously used in Biagini and Frittelli (2008) for the special case of $B = 0$, as explained in Section 2.

We then use this general framework for the case of a random endowment B in Section 3 and prove in Theorem 3.8 a duality result of the type

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x + \int_0^T H_t\, dS_t - B\right)\right] \tag{1.2}$$

$$= \min_{\lambda>0,\ Q\in\mathcal{M}^W} \left\{\lambda x - \lambda Q(B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \| Q^s \|\right\}, \tag{1.3}$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is the convex conjugate of the utility function u, defined by

$$\Phi(y) := \sup_{x\in\mathbb{R}} \{u(x) - xy\}, \tag{1.4}$$

while $\mathcal{M}^W$ is the appropriate set of linear pricing functionals Q, which admit the decomposition $Q = Q^r + Q^s$ into regular and singular parts. The penalty term in the right-hand side of (1.3) is split into the expectation $E[\Phi(\lambda \frac{dQ^r}{dP})]$, associated only with the regular part of Q, and the norm $\| Q^s \|$, associated only with its singular part.

From the previous results Biagini and Frittelli (2008) in the case $B = 0$, we expected the presence of the singular part $\| Q^s \|$, due to the fact that we allow possibly unbounded semimartingales. In general, in the presence of a claim B that is not sufficiently integrable, an additional singular term appears in the formula above from $Q(B) = E_{Q^r}[B] + Q^s(B)$ (as shown explicitly in Section 3.5.1).

Regarding the primal utility maximization problem with random endowment, in Theorem 3.8 we also prove the existence of the optimal solution in a slightly different set than $\{\int_0^T H_t\, dS_t \mid H \in \mathcal{H}^W\}$. As it happens in the literature for $B = 0$, this optimal solution exists under additional assumptions on the utility function u (or equivalent growth conditions on its conjugate), which are introduced in Section 3.3.

Since the most well-studied utility function in the class considered in this paper is the exponential utility, we specialize the duality result for this case in Section 3.5, thereby obtaining a generalization of the results in Bellini and Frittelli (2002), the "Six Authors paper" Delbaen et al. (2002), and Becherer (2003). An interesting example of exponential utility optimization with random endowment, where the singular part shows up, is presented in Section 3.5.1. This example is simple and one period market model, but surprising since it displays a quite different behavior from the locally bounded case, which is thoroughly interpreted.

Armed with the duality result of Theorem 3.8, we address the indifference price of a claim B in Section 4. The Orlicz space duality framework enables us to establish the properties of the indifference price π summarized in Proposition 4.2, including the expected properties of convexity, monotonicity, translation invariance, and volume asymptotics. More interestingly, in (4.3) we provide a new and fairly explicit representation for the indifference price, which is obtained applying recent results from the theory of convex risk measures developed in Biagini and Frittelli (2009). In fact, in Proposition 4.2 it is also shown that the map π, as a convex monotone functional on the Orlicz space $L^{\widehat{u}}$, is continuous and subdifferentiable on the interior of its proper domain $\mathcal{B}$. In Corollary 4.4 we show that when B and the loss control W are sufficiently integrable, the indifference price π also has the Fatou property. The regularity of the map π itself allows then for a very nice, short proof of some bounds on the indifference price $\pi(B)$ of a fixed claim B as a consequence of the Max Formula in Convex Analysis.

The results above extend the existing literature on utility maximization with random endowment when u is finite on the entire real line in several respects. First of all, we do not require the semimartingale S to be locally bounded, and as far as we know ours is the first paper in this direction. Secondly, even though we admit price processes represented by general semimartingale, our assumptions on the claim B are weaker than those assumed in the literature for the locally bounded case—a nice consequence of the selection of the Orlicz space duality.

While the notion of the indifference price was introduced in 1989 by Hodges and Neuberger (1989), the analysis of its dual representation in terms of (local) martingale measures was performed in the late 1990s. It started with Frittelli (2000) and Bellini and Frittelli (2002), and was considerably expanded by Delbaen et al. (2002) and, in a dynamic context, by El Karoui and Rouge (2000) and Becherer (2003). An extensive survey of the recent literature on this topic can be found in Carmona (2009). The classical approach of Convex Analysis—basically the Fenchel-Moreau Theorem—was first applied in Frittelli and Rosazza Gianin (2002) to deduce the dual representation of convex risk measures on L^p spaces. Based on the duality results proven in Frittelli (2000), in Frittelli and Rosazza Gianin (2002) it is also shown that, for the exponential utility function, the indifference price of a bounded claim defines—except for the sign—a convex risk measure. In recent years this connection has been deeply investigated by many authors (see Barrieu and El Karoui 2008 and the references therein).

Finally, let us remark that Owen and Zitkovich (2009) also consider unbounded claims B for a general utility u defined on $\mathbb{R}$, but for locally bounded semimartingales. Moreover, their assumption 1.6 on the claim is also of a different type, since it is a joint condition on B and the admissible strategies. This condition is not easy to verify in practice, since it requires the prior knowledge of the dual measures. Also, for economic reasons, we believe that it is better to state the conditions on the claim only in terms of the compatibility with the utility function.

2. THE SETUP FOR UTILITY MAXIMIZATION

We briefly recall the setup of Biagini and Frittelli (2008) for the utility maximization problem, namely the extended class of admissible strategies and the Orlicz duality. The same framework is used in the next section for the optimization problem in the presence of a random endowment.

2.1. Admissible Strategies

Given a fixed, nonnegative random variable $W \in \mathcal{F}_T$, the domain of optimization for the primal problem (1.2) is the following set of *W-admissible strategies*

$$\mathcal{H}^W := \left\{ H \in L(S) \mid \exists c > 0 \text{ such that } \int_0^t H_s\, dS_s \geq -cW, \forall t \in [0, T] \right\}, \tag{2.1}$$

where $L(S)$ denotes the class of predictable, S-integrable processes. In other words, the random variable W controls the losses in trading of the wealth process $\int H dS$.

In order to build a reasonable utility maximization, the loss control W should satisfy two extra assumptions. The first depends only on the market S and guarantees that the set of W-admissible strategies is rich enough for trading purposes.

DEFINITION 2.1. A random variable $W \geq 1$ is suitable (for the process S) if for each $i = 1, \ldots, d$, there exists a process $H^i \in L(S^i)$ such that

$$P\left(\left\{\omega \mid \exists t \geq 0 \text{ such that } H_t^i(\omega) = 0\right\}\right) = 0 \tag{2.2}$$

and

$$\left| \int_0^t H_s^i\, dS_s^i \right| \leq W, \quad \forall t \in [0, T]. \tag{2.3}$$

The second condition depends exclusively on the utility function and reflects to what extent the investor accepts the risk of large losses:

DEFINITION 2.2. A positive random variable W is strongly compatible with the utility function u if

$$E[u(-\alpha W)] > -\infty \text{ for all } \alpha > 0 \tag{2.4}$$

and it is compatible with u if

$$E[u(-\alpha W)] > -\infty \text{ for some } \alpha > 0. \tag{2.5}$$

When S is locally bounded, $W = 1$ is automatically suitable and compatible (see Biagini and Frittelli 2005, proposition 1), and we recover the familiar set of admissible trading strategies, namely those with wealth process bounded below by a constant. For the non locally bounded case, the existence of a suitable and compatible loss variable is not automatically guaranteed, and is related to integrability restrictions on the jumps of the semimartingale S. From now on, we assume the existence of a fixed suitable and compatible W and work with the associated class of strategies $\mathcal{H}^W$.

The first step to apply duality arguments to problem (1.2) is to rewrite it in terms of an optimization over random variables, as opposed to an optimization over stochastic processes. To this end, we define the set of terminal values obtained from W-admissible trading strategies as

$$K^W = \left\{ \int_0^T H_t\, dS_t \mid H \in \mathcal{H}^W \right\}, \tag{2.6}$$

and consider the reformulated primal problem

$$\sup_{k\in K^W} E[u(x+k)]. \tag{2.7}$$

The next step is to identify an appropriate dual system. Classically, the system (L^∞, ba) has been successfully used when dealing with locally bounded traded assets. In order to accommodate more general markets and inspired by the compatibility conditions above, in the next section we argue instead for the use of an appropriate Orlicz spaces duality, naturally induced by the utility function.

2.2. The Orlicz Space Framework

Orlicz spaces are generalizations of L^p spaces, and a reference book on the topic is Rao and Ren (1991). Their use in mathematical finance is quite new, as it was first introduced by Biagini (2008) in the context of utility maximization and then considerably expanded in Biagini and Frittelli (2008). The key point is that the function $\widehat{u}:\mathbb{R}\to[0,+\infty)$

$$\widehat{u}(x) = -u(-|x|)+u(0),$$

is a Young function (i.e., even, convex, and finite in a neighborhood of 0) and can therefore be used to define the corresponding Orlicz space

$$L^{\widehat{u}}(\Omega,\mathcal{F},P) := \{f\in L^0 \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for some } \alpha>0\},$$

which is a Banach space when equipped with the Luxemburg norm

$$N_{\widehat{u}}(f) = \inf\left\{c>0 \mid E\left[\widehat{u}\left(\frac{f}{c}\right)\right]\le 1\right\}. \tag{2.8}$$

Given the standing assumptions on the utility u, it consists of integrable random variables and it contains all the bounded variables, that is $L^\infty\subseteq L^{\widehat{u}}\subseteq L^1$. The elements in the *Morse subspace* $M^{\widehat{u}}$ of $L^{\widehat{u}}$ satisfy a stronger integrability condition

$$M^{\widehat{u}} := \{f\in L^{\widehat{u}} \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for all } \alpha>0\}.$$

The space $M^{\widehat{u}}$ also contains L^∞ (see, e.g., Rao and Ren 1991, chapter III for extra details and closure properties), but is in general strictly contained in $L^{\widehat{u}}$. For example, for an exponential utility $u(x)=-e^{-x}$, we have $\widehat{u}(x)=e^{|x|}-1$, and $L^{\widehat{u}}$ is the space of random variables with a finite absolute exponential moment, while $M^{\widehat{u}}$ is the subspace of those random variables with all finite absolute exponential moments. A random variable X exponentially distributed with parameter $\lambda>0$: $X\sim\mathcal{E}(\lambda)$ is in $L^{\widehat{u}}$ but not in $M^{\widehat{u}}$.

Moreover a positive random variable W is strongly compatible (resp. compatible) with the utility function u if and only if $W\in M^{\widehat{u}}$ (resp. $W\in L^{\widehat{u}}$). When W belongs to either of these spaces, the *negative part* of any element in $k\in K^W$ belongs to the same space, since the control $k\ge -cW$ implies $k^-\le cW$.

The convex conjugate of $\widehat{u}$, called the *complementary* Young function in the theory of Orlicz spaces, is denoted here by $\widehat{\Phi}$, since it admits a representation in terms of Φ

$$\widehat{\Phi}(y) = (\Phi(|y|)-\Phi(\beta))\,1_{\{|y|>\beta\}}, \tag{2.9}$$

where $\beta \geq 0$ is the right derivative of $\widehat{u}$ at 0. It then follows that $\widehat{\Phi}$ is also a Young function, which induces the Orlicz space $L^{\widehat{\Phi}}$. As before, $L^\infty \subseteq L^{\widehat{\Phi}} \subseteq L^1$ and in addition $L^{\widehat{\Phi}}$ is the topological dual of the Morse space $M^{\widehat{u}}$,

$$(M^{\widehat{u}})^* = L^{\widehat{\Phi}}. \tag{2.10}$$

The description of the topological dual for the Orlicz space $L^{\widehat{u}}$ is more demanding. Each element $Q \in (L^{\widehat{u}})^*$ admits a unique decomposition $Q = Q^r + Q^s$ (see, e.g., Rao and Ren 1991, corollary IV.2.9), where the *regular* part Q^r is a signed measure and thus can be represented by its density $\frac{dQ^r}{dP}$, which belongs to $L^{\widehat{\Phi}}$, while the *singular* part Q^s satisfies

$$Q^s(f) = 0, \forall f \in M^{\widehat{u}}. \tag{2.11}$$

2.3. Dual Variables

The dual system $(L^{\widehat{u}}, (L^{\widehat{u}})^*)$ is now ready, but the domain of the utility maximization K^W is not yet a subset of $L^{\widehat{u}}$. The standard trick is to consider

$$C^W = \left(K^W - L^0_+\right) \cap L^{\widehat{u}},$$

instead of K^W. The cone C^W corresponds to random variables that can be super-replicated by admissible strategies in $\mathcal{H}^W$ and belong to $L^{\widehat{u}}$, i.e., satisfy the same type of integrability condition as W. The polar cone of C^W is the domain of the auxiliary dual problem and it is defined

$$(C^W)^0 := \left\{Q \in (L^{\widehat{u}})^* \mid Q(f) \leq 0, \forall f \in C^W\right\}. \tag{2.12}$$

Since $(-L^{\widehat{u}}_+) \subseteq C^W$, all functionals in $(C^W)^0$ are positive. The subset of normalized functionals in $(C^W)^0$ is

$$\mathcal{M}^W := \{Q \in (C^W)^0 \mid Q(\mathbf{1}_\Omega) = 1\}. \tag{2.13}$$

This normalization condition reduces to $Q^r(\mathbf{1}_\Omega) = 1$, since Q^s vanishes on any bounded random variable from (2.11). In other words, *the regular part of any element in $\mathcal{M}^W$ is a true probability measure with density in $L^{\widehat{\Phi}}_+$*. It was shown in Biagini and Frittelli (2008, proposition 19), that the subset of true probabilities in $\mathcal{M}^W$ has nice properties. It is independent of the specific W and has a clear financial interpretation

$$\{Q \in \mathcal{M}^W \mid Q^s = 0\} = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}, \tag{2.14}$$

where $\mathbb{M}_\sigma := \{Q \ll P \mid S \text{ is a } \sigma- \text{ martingale w.r.t. } Q\}$ consists of all the P-absolutely continuous σ-martingale measures for S. When S is continuous or locally bounded, $\mathbb{M}_\sigma$ are simply the local martingale measures for S. For the relevance of σ-martingales in mathematical finance as pricing measures when S is non locally bounded, the reader is referred to Delbaen and Schachermayer (1998).

3. UTILITY OPTIMIZATION WITH RANDOM ENDOWMENT

The main result in this section is Theorem 3.8, establishing a duality result for utility optimization with a random endowment B. When $B = 0$ such duality is proved in Biagini and Frittelli (2008), and the goal of this section is to show that the same arguments still apply in the presence of B.

3.1. Conditions on the Claim

Consider then the problem

$$\sup_{H \in \mathcal{H}^W} E\left[u\left(x + \int_0^T H_t \, dS_t - B\right)\right]. \tag{3.1}$$

Set $x = 0$, since as the case with nonnull initial capital can be recovered by replacing B with $(B - x)$. In view of the replacement of terminal wealths $\int_0^T H_t \, dS_t \in K^W$ by random variables $f \in C^W \subset L^{\widehat{u}}$, we introduce the following:

DEFINITION 3.1. Let $\mathcal{B}$ be the set of claims $B \in L^{\widehat{u}}$ satisfying

$$E[u(-(1+\epsilon)B^+)] > -\infty, \text{ for some } \epsilon > 0. \tag{3.2}$$

The two requirements above can both be regarded as well-posedness conditions. If $B \in L^{\widehat{u}}$, then $E[u(f - B)] < +\infty$ for all f in the maximization domain $C^W \subset L^{\widehat{u}}$, because

$$E[u(f - B)] \overset{Jensen}{\leq} u(E[f - B]) < +\infty,$$

since $(f - B) \in L^{\widehat{u}} \subseteq L^1$. The requirement (3.2) is an Orlicz type condition on the seller's loss B^+, enabling the application of variational arguments for the maximization. In particular, such requirement implies that $E[u(-B)] > -\infty$, so that B does not lead to prohibitive punishments when the seller chooses the trading strategy $H \equiv 0 \in \mathcal{H}^W$, which in turn implies that $\sup_{f \in C^W} E[u(f - B)] > -\infty$. Additional properties of the set $\mathcal{B}$ are established in Lemma 4.5.[1]

3.2. A First Dual Formula

When $B \in \mathcal{B}$, a monotonicity argument shows that optimizing over the cone C^W leads to the same expected utility as optimizing over the set of terminal wealths K^W

$$\sup_{k \in K^W} E[u(k - B)] = \sup_{f \in C^W} E[u(f - B)]. \tag{3.3}$$

It is well known in the literature that the primal problem (3.3) may not attain its maximum on C^W (or on K^W). The duality approach shifts the analysis over an auxiliary minimization problem (dual problem), which attains the minimum. The dual problem is used to characterize a posteriori a domain over which the maximizer of the primal

[1] The main duality result in Section 3.4 can be established under more general conditions on the negative part of the claim B, at the cost of having to define appropriate extensions for linear functionals on $L^{\widehat{u}}$, as we did in previously circulated versions of this paper.

problem is attained. This is the content of the next subsection, but first we show a dual formula.

Let $I_u^B := E[u(f-B)]$ be a concave integral functional on $L^{\widehat{u}}$ with values in $[-\infty, +\infty)$. The next lemma establishes that I_u^B satisfies the conditions necessary to invoke Fenchel's Duality Theorem.

LEMMA 3.2. *If $B \in \mathcal{B}$, the functional I_u^B is norm continuous on the interior of its proper domain* $\mathrm{Dom}(I_u^B) = \{f \in L^{\widehat{u}} \mid I_u^B(f) \in \mathbb{R}\}$. *As a consequence, there exists a norm continuity point of I_u^B that belongs to C^W.*

Proof. Since $I_u^B < +\infty$, then I_u^B is proper, monotone, and concave. The first statement thus follows from the Extended Namioka-Klee Theorem (see Ruszczynski and Shapiro 2006; Biagini and Frittelli 2009). If g belongs to $\mathcal{S}_1$, the open unit ball in $L^{\hat{u}}$, then $E[u(g)]$ is finite and (3.2) implies

$$(+\infty >)E\left[u\left(-B+\frac{\epsilon}{1+\epsilon}g\right)\right] \geq E\left[u\left(-B^{+}+\frac{\epsilon}{1+\epsilon}g\right)\right]$$
$$\geq \frac{E[u(-(1+\epsilon)B^{+})]+\epsilon E[u(g)]}{1+\epsilon} > -\infty$$

hence

$$\frac{\epsilon}{1+\epsilon}\mathcal{S}_1 \subset \mathrm{Dom}(I_u^B). \tag{3.4}$$

Therefore, any element of $\frac{\epsilon}{2(1+\epsilon)}\mathcal{S}_1 \cap (-L_+^{\widehat{u}})$ is a continuity point for I_u^B that belongs to C^W. □

Here is a first dual formula, valid under the assumption that bliss utility cannot be reached.

LEMMA 3.3. *Suppose that $B \in \mathcal{B}$ and that* $\sup_{f\in C^W} E[u(f-B)] < u(+\infty)$. *Then the following dual formula holds:*

$$\sup_{f\in C^W} E[u(f-B)] = \min_{\lambda>0,\, Q\in\mathcal{M}^W}\left\{\lambda(Q(-B)+\|Q^s\|)+E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]\right\}. \tag{3.5}$$

Proof. From Fenchel's Duality Theorem, we have

$$\sup_{f\in C^W} I_u^B(f) = \min_{Q\in(C^W)^0}\left(I_u^B\right)^*(Q),$$

where $(I_u^B)^*(Q) := \sup_{f\in L^{\widehat{u}}}\{I_u^B(f) - Q(f)\}$ is the convex conjugate of I_u^B. A result by Kozek (1979) shows that the convex conjugate has the representation $(I_u^B)^*(Q) = Q(-B) + \|Q^s\| + E[\Phi(\frac{dQ^r}{dP})]$. Recalling that $\mathcal{M}^W$ is the set of normalized elements in $(C^W)^0$, a simple reparameterization of the polar cone leads to the minimization over $\mathcal{M}^W$ modulo a scaling factor λ. The no bliss condition $\sup_{f\in C^W} I_u^B(f) < u(+\infty) = \Phi(0)$ ensures that the minimizer cannot be null, hence the thesis follows. □

3.3. The Dual Optimization and a New Primal Domain

The expectation term appearing in the minimization problem in Lemma 3.3 leads us to consider the set

$$\mathcal{L}_\Phi = \left\{ Q \ll P \mid E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] < +\infty, \text{ for some } \lambda > 0 \right\}.$$

Despite the possibility that $\Phi(0) = +\infty$, this is a convex set, as a consequence of the convexity of the function $(z, k) \to z\Phi(\frac{1}{z}k)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, which has been pointed out by Schied and Wu (2005, section 3).

It is sometimes necessary to guarantee that the expectation above is finite regardless of the value of the scaling factor λ. When that is the case, we will require the following:

ASSUMPTION 3.4. *The utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable, and*

$$\lim_{x \downarrow -\infty} u'(x) = +\infty, \ \lim_{x \uparrow \infty} u'(x) = 0 \quad (\textit{Inada conditions}) \tag{3.6}$$

Moreover, for any probability $Q \ll P$, the conjugate function Φ satisfies

$$E\left[\Phi\left(\frac{dQ}{dP}\right)\right] < +\infty \textit{ iff } \quad E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] < +\infty \textit{ for all } \lambda > 0. \tag{3.7}$$

Condition (3.7) above is a joint condition on the probabilistic model and the preferences via the conjugate Φ. A detailed discussion on relationship of (3.7) with the condition of Reasonable Asymptotic Elasticity introduced by Schachermayer (2001) can be found in Biagini and Frittelli (2005, 2008).

A general strategy for tackling minimization problems of the form appearing in (3.5) is to consider the minimizations over λ and over Q separately. The two Propositions below are a generalization of the corresponding ones in Biagini and Frittelli (2008). In the first one we fix λ and explore the consequences of optimality in Q:

PROPOSITION 3.5. *Let $B \in \mathcal{B}$ and suppose that Assumption 3.4 holds. Fix $\lambda > 0$ and suppose that $\mathcal{N} \subseteq (L^{\widehat{u}})^*_+$ is a convex set such that for any $Q \in \mathcal{N}$ we have $Q^r \in \mathcal{L}_\Phi$. If $Q_\lambda \in \mathcal{N}$ is optimal for*

$$\inf_{Q \in \mathcal{N}} \left\{ E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda(Q(-B) + \| Q^s \|) \right\} < +\infty \tag{3.8}$$

then, $\forall Q \in \mathcal{N}$

$$E_{Q^r_\lambda}\left[\Phi'\left(\lambda \frac{dQ^r_\lambda}{dP}\right)\right] + Q_\lambda(-B) + \| Q^s_\lambda \| \leq E_{Q^r}\left[\Phi'\left(\lambda \frac{dQ^r_\lambda}{dP}\right)\right] + Q(-B) + \| Q^s \|. \tag{3.9}$$

Next we fix Q and explore the consequences of optimality in λ.

PROPOSITION 3.6. *Under Assumption 3.4, if Q is a probability measure in $\mathcal{L}_\Phi$, then for all $c \in \mathbb{R}$ the optimizer $\lambda(c; Q)$ of*

$$\min_{\lambda > 0} \left\{ E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] + \lambda c \right\} \tag{3.10}$$

is the unique positive solution of the first order condition

$$E\left[\frac{dQ}{dP}\Phi'\left(\lambda\frac{dQ}{dP}\right)\right]+c=0. \tag{3.11}$$

Moreover, the random variable $g^* := -\Phi'\left(\lambda(c;Q)\frac{dQ}{dP}\right)$ *belongs to the set* $\{g \in L^1(Q) \mid E_Q[g]=c\}$, *satisfies* $u(g^*) \in L^1(P)$, *and*

$$\begin{aligned}\min_{\lambda>0}\left\{E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]+\lambda c\right\} &= \sup\{E[u(g)] \mid g \in L^1(Q) \quad \textit{and} \\ &\quad E_Q[g]\le c\} = E[u(g^*)] < u(+\infty).\end{aligned} \tag{3.12}$$

For a fix $Q \in \mathcal{M}^W$ such that $Q^r \in \mathcal{L}_\Phi$, set $c = Q(-B) + \|Q^s\|$. Then the message of the above proposition is that the minimization of the dual function in Lemma 3.3 with respect to $\lambda > 0$ leads to the same value of a utility maximization over integrable functions satisfying $E_{Q^r}[g] \le Q(-B) + \|Q^s\|$. Setting $g = (f - B)$ to be consistent with the notation of Lemma 3.3, this leads us to define the following set of functionals and corresponding domain for utility maximization.

DEFINITION 3.7. Let

$$\mathcal{M}^W_\Phi := \{Q \in \mathcal{M}^W \mid Q^r \in \mathcal{L}_\Phi\} \tag{3.13}$$

and

$$K^W_B := \{f \in L^0 \mid f \in L^1(Q^r),\ E_{Q^r}[f] \le Q^s(-B) + \|Q^s\|, \forall Q \in \mathcal{M}^W_\Phi\}, \tag{3.14}$$

with the corresponding optimization problem

$$U^W_B := \sup_{f\in K^W_B} E[u(f-B)]. \tag{3.15}$$

Lemma 3.3 ensures that $\mathcal{M}^W_\Phi$ is not empty, since the minimum is attained on functionals such that $Q^r \in \mathcal{L}_\Phi$.

3.4. The Main Duality Result

THEOREM 3.8. *Let* $u:\mathbb{R}\to\mathbb{R}$ *be an increasing and concave function satisfying* $\lim_{x\to-\infty} u(x) = -\infty$. *If* $B \in \mathcal{B}$ *and*

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(\int_0^T H_t\,dS_t - B\right)\right] < u(+\infty), \tag{3.16}$$

then $\mathcal{M}^W_\Phi$ *is not empty and*

$$\begin{aligned}\sup_{H\in\mathcal{H}^W} E\left[u\left(\int_0^T H_t\,dS_t - B\right)\right] &= U^W_B \\ &= \min_{\lambda>0,\ Q\in\mathcal{M}^W_\Phi}\left\{\lambda Q(-B) + E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]+\lambda\|Q^s\|\right\}.\end{aligned} \tag{3.17}$$

The minimizer λ_B *is unique, while the minimizer* Q_B *is unique only in the regular part* $Q^r_B \neq 0$. *Suppose in addition that the utility satisfies Assumption 3.4. Then the maximum* *is attained over* K_B^W *and the unique maximizer is*

$$f_B = -\Phi'\left(\lambda_B \frac{dQ_B^r}{dP}\right) + B. \tag{3.18}$$

The relation between primal and dual optimizers is given by

$$E_{Q_B^r}[f_B] = Q_B^s(-B) + \| Q_B^s \|. \tag{3.19}$$

Proof. The chain of equalities in (3.17) follows from (3.3) and from

$$\sup_{f \in C^W} E[u(f - B)] = U_B^W = \min_{\lambda>0,\, Q\in\mathcal{M}_\Phi^W} \left\{ \lambda\, Q(-B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \| Q^s \| \right\}. \tag{3.20}$$

To establish (3.20), fix an $f \in C^W$ such that $E[u(f - B)] > -\infty$. As shown in Biagini and Frittelli (2008, lemma 17), the norm $\| Q^s \|$ of the singular part Q^s of any $Q \in (L^{\widehat{u}})^*_+$ satisfies

$$\| Q^s \| = \sup_{\{g \in L^{\widehat{u}} \mid E[u(-g)] > -\infty\}} Q^s(g), \tag{3.21}$$

and since $(B - f) \in L^{\widehat{u}}$, we have that $\| Q^s \| \geq Q^s(B - f)$. For any $Q \in \mathcal{M}^W$, and in particular for those in $\mathcal{M}_\Phi^W$, $Q(f) \leq 0$ so $Q(f - B) \leq Q(-B)$. Canceling $E_{Q^r}(-B)$ from both sides, $E_{Q^r}(f) + Q^s(f - B) \leq Q^s(-B)$ so that

$$E_{Q^r}(f) \leq Q^s(-B) + Q^s(B - f) \leq Q^s(-B) + \| Q^s \|$$

and $f \in K_B^W$, which implies that $\sup_{f \in C^W} E[u(f - B)] \leq U_B^W$. Now, fix an $f \in K_B^W$, a positive λ and a $Q \in \mathcal{M}_\Phi^W$. Fenchel's inequality then gives

$$u(f - B) \leq \lambda \frac{dQ^r}{dP}(f - B) + \Phi\left(\lambda \frac{dQ^r}{dP}\right).$$

Taking expectations and using (3.14),

$$E[u(f - B)] \leq \lambda E\left[\frac{dQ^r}{dP}(f - B)\right] + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] \leq \lambda (Q(-B) + \| Q^s \|) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right]$$

so

$$U_B^W \leq \min_{\lambda>0,\, Q\in\mathcal{M}_\Phi^W} \left\{ \lambda\, Q(-B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \| Q^s \| \right\}.$$

Then (3.20) follows from Lemma 3.3.

If a dual minimizer Q had $Q^r = 0$, then the dual minimum would be $\Phi(0) + \lambda_B(Q^s \times (-B) + \| Q^s \|) \geq u(+\infty)$, since $\| Q^s \| \geq Q^s(B)$ from (3.21) and $\Phi(0) = u(\infty)$. This would contradict condition (3.16). Uniqueness of λ_B and Q_B^r follow from strict convexity of the dual objective function in λ and Q^r. However, the dual objective function is not strictly convex in Q^s, since the norm is additive on positive singular functionals: $\| Q_1^s + Q_2^s \| = \| Q_1^s \| + \| Q_2^s \|$, see Biagini and Frittelli (2008, proposition 11). Therefore the optimal singular functional might not be unique.

Under Assumption 3.4, the expression for f_B and its relation with Q_B can be derived from a combination of the results in Propositions 3.5 and 3.6 by observing that any minimizer Q^B is obtained as the minimizer of

$$\min_{Q\in\mathcal{M}_\Phi^W}\left\{\lambda_B Q(-B)+E\left[\Phi\left(\lambda_B\frac{dQ^r}{dP}\right)\right]+\lambda_B\|Q^s\|\right\}.$$

□

COROLLARY 3.9. *If both W and B are in the Morse subspace $M^{\widehat{u}}$, then $\mathcal{M}_\Phi^W$ can be replaced in (3.17) by $\mathbb{M}_\sigma\cap\mathcal{L}_\Phi$ and no singular term appears in the dual problem.*

Proof. If $W\in M^{\widehat{u}}$, then the regular component Q^r of $Q\in\mathcal{M}^W$ is already in $\mathcal{M}^W$ (see Biagini and Frittelli 2008, lemma 41). If B is in $M^{\widehat{u}}\subseteq\mathcal{B}$ as well, then $Q_s(B)=0$. Since $\|Q^s\|\geq 0$, the minimum of the dual problem must be achieved on the set $\mathbb{M}_\sigma\cap\mathcal{L}_\Phi$. Therefore, a posteriori one can rewrite the dual as a minimization over these probabilities only. □

The next proposition gives a priori bounds for this singular contribution $Q^s(-B)$ appearing in (3.17)

PROPOSITION 3.10. *For any $B\in\mathcal{B}$, let*

$$L:=\sup\{\beta>0\mid E[\widehat{u}(\beta B^+)]<+\infty\}\quad\textit{and}\quad l:=\sup\{\alpha>0\mid E[\widehat{u}(\alpha B^-)]<+\infty\}.$$

Then, for any fixed $Q\in\mathcal{M}_\Phi^W$,

$$-\frac{1}{L}\|Q^s\|\leq Q^s(-B)\leq\frac{1}{l}\|Q^s\| \tag{3.22}$$

and in particular we recover again $Q^s(B)=0$ when $B\in M^{\widehat{u}}$.

Proof. It follows from (3.2) that $L\geq 1+\epsilon$. For any $b<L$, (3.21) gives $\|Q^s\|\geq b\,Q^s(B^+)$ and therefore

$$Q^s(-B)\geq -Q^s(B^+)\geq-\frac{1}{b}\|Q^s\|,$$

hence $Q^s(-B)\geq-\frac{1}{L}\|Q^s\|$. To prove the right inequality in (3.22), observe that $l>0$ and that $-\alpha B^-\in\mathrm{Dom}(I_u)$ for any $\alpha<l$. Therefore

$$Q^s(-B)\leq Q^s(B^-)=\frac{1}{\alpha}Q^s(\alpha B^-)\leq\frac{1}{\alpha}\|Q^s\|\ \text{ for all }\alpha<l.$$ □

The result in Theorem 3.8 does not guarantee in full generality that the optimal random variable $f_B\in K_B^W$ can be represented as terminal value from an investment strategy in $L(S)$, that is, $f_B=\int_0^T H_t\,dS_t$. The next proposition presents a partial result in this direction.

PROPOSITION 3.11. *Suppose that Assumption 3.4 holds and $B\in\mathcal{B}$. If $Q_B^s=0$ and $Q_B^r\sim P$, then f_B can be represented as terminal wealth from a suitable strategy H.*

Proof. When the optimal Q_B has zero singular part, then it is a sigma martingale measure with finite entropy, according to (2.14). Therefore, a posteriori the dual problem (3.17) can be reformulated as a minimum over those $Q\in\mathbb{M}_\sigma$ with finite entropy. In this simplified setup, one can show exactly as in Biagini and Frittelli (2005, theorem 4, theorem 1(d)) that the optimal f_B can be represented as terminal wealth from a suitable strategy H. □

3.5. Exponential Utility

For an exponential utility function $u(x) = -e^{-\gamma x}$, $\gamma > 0$, we have $\Phi(y) = \frac{y}{\gamma}\log\frac{y}{\gamma} - \frac{y}{\gamma}$ and $\widehat{u}(x) = e^{\gamma|x|} - 1$. As already mentioned in Section 2.2, $M^{\widehat{u}}$ consists of those random variables that have *all* the (absolute) exponential moments finite, while the larger space $L^{\widehat{u}}$ corresponds to random variables that have *some* finite exponential moment.

An exponential utility satisfies Assumption 3.4, so the duality results follow directly as a corollary of Theorem 3.8.

COROLLARY 3.12. *Suppose B satisfies*

$$E[e^{(\gamma+\epsilon)B}] < +\infty \quad \textit{and} \quad E[e^{-\epsilon B}] < +\infty \quad \textit{for some} \quad \epsilon > 0.$$

If

$$\sup_{H\in\mathcal{H}^W} E[-e^{-\gamma(\int_0^T H_t\, dS_t - B)}] < 0, \tag{3.23}$$

then $\mathcal{M}_\Phi^W$ is not empty and

$$\sup_{H\in\mathcal{H}^W} E[-e^{-\gamma(\int_0^T H_t\, dS_t - B)}] = -\exp\left\{-\min_{Q\in\mathcal{M}_\Phi^W}(H(Q^r\,|\,P) + \gamma\, Q(-B) + \gamma\,\|\, Q^s\|)\right\}, \tag{3.24}$$

where $H(Q^r\,|\,P) = E[\frac{dQ^r}{dP}\log(\frac{dQ^r}{dP})]$ denotes the relative entropy of Q^r with respect to P. The minimizer $Q_B \in \mathcal{M}_\Phi^W$ is unique only in the regular part Q^r_B. In addition,

$$\sup_{H\in\mathcal{H}^W} E\Big[-e^{-\gamma(\int_0^T H_t\, dS_t - B)}\Big] = E[-e^{-\gamma(f_B - B)}],$$

the optimal claim is

$$f_B = -\frac{1}{\gamma}\ln\left(\frac{\lambda_B}{\gamma}\frac{dQ^r_B}{dP}\right) + B,$$

where $\lambda_B = \gamma\exp(H(Q^r_B|\,P) + \gamma\, Q_B(-B) + \gamma\,\|\, Q^s_B\|) = -\frac{1}{\gamma}U_B^W$, and it satisfies

1. *$f_B \in L^1(Q^r)$, $E_{Q^r}[f_B] \le Q^s(-B) + \|\, Q^s\|$ for all $Q \in \mathcal{M}_\Phi^W$ (i.e., it belongs to K_B^W)*
2. *$E_{Q^r_B}[f_B] = Q^s_B(-B) + \|\, Q^s_B\|$*

If W and B have all the exponential (absolute) moments finite, then $\mathcal{M}_\Phi^W$ can be replaced by the "classic" set of probabilities $Q \in \mathbb{M}_\sigma$ that have finite relative entropy, i.e., $E[\frac{dQ}{dP}\ln(\frac{dQ}{dP})] < +\infty$, and no singular term appears in (3.24).

Proof. The conditions on B are exactly those in Theorem 3.8, adapted to the exponential case. So, from Theorem 3.8

$$\sup_{H\in\mathcal{H}^W} E\left[-e^{-\gamma(\int_0^T H_t\,dS_t - B)}\right]$$
$$= \min_{\lambda>0,\ Q\in\mathcal{M}_\Phi^W}\left\{\lambda Q(-B) + E\left[\frac{\lambda}{\gamma}\frac{dQ^r}{dP}\log\left(\frac{\lambda}{\gamma}\frac{dQ^r}{dP}\right) - \frac{\lambda}{\gamma}\frac{dQ^r}{dP}\right] + \lambda\|Q^s\|)\right\}$$

and an explicit minimization over $\lambda > 0$ leads to the duality formula (3.24). The remaining assertions follow as in the proof of Theorem 3.8. □

3.5.1 Example with Nonzero Singular Parts. Consider a one period model with $S_0 = 0$ and $S_1 = YZ$ where Y is an exponential random variable with density $f(y) = e^{-y}$, $y \geq 0$ and Z is a discrete random variable taking the values $\{1, -\frac{1}{2}, \ldots, \frac{1}{n} - 1, \ldots\}$. Assume that Y and Z are independent and let $p_1 := P(Z = 1) > 0$ and $p_n := P(Z = \frac{1}{n} - 1) > 0$, $n \geq 2$, be the probability distribution of Z. For an investor with exponential utility $u(x) = -e^{-x}$, it is clear that the random variable $W = 1 + Y$ is suitable and compatible. Suppose now that $B = \alpha(Y, Z)$, where α is a bounded Borel function, so that the seller of the claim B faces the problem

$$\sup_{h\in\mathbb{R}} E[-e^{-hS_1+B}] = \sup_{h\in\mathbb{R}} E[-e^{-hZY+\alpha(Y,Z)}].$$

A necessary condition for the expectations to be finite is that $-1 < h \leq 1$. In fact, using the probability distribution of Z,

$$E[-e^{-hZY+\alpha(Y,Z)}] = p_1 E[-e^{-hY+\alpha(Y,1)}] + \sum_{n\geq 2} p_n E[-e^{-hz_n Y+\alpha(Y,z_n)}]$$
$$= p_1\int_0^{+\infty} e^{-hy+\alpha(y,1)}e^{-y}dy + \sum_{n\geq 2} p_n \int_0^{+\infty} e^{-(\frac{1}{n}-1)hy+\alpha(y,\frac{1}{n}-1)}e^{-y}dy.$$

If the expectation on the LHS is finite, then necessarily the one-dimensional integrals on RHS converge. Since α is bounded, these integrals converge iff $-1 < h \leq 1$.

The function to be optimized, $g_B(h) := E[-e^{-hS_1+B}]$, has a formal derivative given by

$$g'_B(h) = E[S_1 e^{-hS_1+B}] = p_1 E[Ye^{-hY+\alpha(Y,1)}] + \sum_{n\geq 2} p_n z_n E[Ye^{-hz_n Y+\alpha(Y,z_n)}].$$

Since $-1 < z_n < 0$ for $n \geq 2$,

$$g'_B(h) \geq p_1 E[Ye^{-Y+\alpha(Y,1)}] - \sum_{n\geq 2} p_n E[Ye^{-z_n Y+\alpha(Y,z_n)}].$$

When $p_n \to 0$ sufficiently fast, this expression is not only well defined but strictly positive. To fix the ideas, we show how to compute such p_n. Integration by parts of the terms $E[Ye^{-Y}]$, $E[Ye^{-z_n Y}]$ gives $\frac{1}{4}$ and $\frac{1}{(-z_n-1)^2} = n^2$, respectively, so that

$$g'_B(h) \geq p_1\frac{e^{-\|B\|_\infty}}{4} - \sum_{n\geq 2} p_n e^{+\|B\|_\infty} n^2 = \frac{e^{-\|B\|_\infty}}{4}\left[p_1 - \sum_{n\geq 2} p_n 4e^{+2\|B\|_\infty} n^2\right],$$

where $\|B\|_\infty$ denotes the sup-norm of the bounded $B = \alpha(Y, Z)$. Fix $k \geq 2$ and for $n \geq 2$ define $p_n := \frac{1}{4e^{+2\|B\|_\infty} n^2 2^{n-1+k}}$. Accordingly, $p_1 := (1 - \sum_{n\geq 2} p_n) > 1 - \frac{1}{2^k}$ so that

$$\left[p_1 - \sum_{n\geq 2} p_n 4e^{+2\|B\|_\infty} n^2\right] = p_1 - \frac{1}{2^k} > 1 - \frac{1}{2^{k-1}} > 0.$$

Under such choice for the distribution of Z, it is guaranteed that g'_B is indeed the true derivative of g_B and that $0 < g'_B(h) < \infty$ for all $-1 < h \leq 1$. Therefore, the function $g_B(h)$ is strictly increasing and attains its maximum at $h = 1$. But this implies that $\sup_{h\in\mathbb{R}} E[-e^{-hS_1+B}] = E[-e^{-S_1+B}]$, so that the optimizer for the primal problem is $f_B = S_1$. From the identity $u'(f_B - B) = \lambda_B \frac{dQ^r_B}{dP}$, the optimizer for the dual problem has a regular part given by

$$\frac{dQ^r_B}{dP} = \frac{e^{-S_1+B}}{E[e^{-S_1+B}]}. \tag{3.25}$$

Using (3.25) to calculate the expectation of f_B with respect to Q^r_B and the fact that $Q^s_B(-B) = 0$ (since B is bounded), we conclude from (3.19) that

$$\|Q^s_B\| = E_{Q^r_B}[f_B] = \frac{E[S_1 e^{-S_1+B}]}{E[e^{-S_1+B}]} = \frac{g'_B(1)}{E[e^{-S_1+B}]} > 0,$$

which implies that $Q^s_B \neq 0$.

Observe that these probabilities p_n also guarantee that the utility maximization problem without the claim B leads to a function $g_0(h)$ with the same domain as g_B. Also, $0 < g'_0 < \infty$. In particular, the maximum of $E[-e^{-hS_1}]$ is still attained at $h = 1$, which implies that the optimizers f_0 and f_B for the primal problem with and without the claim coincide. This means that the investor does not use the underlying market to hedge the claim, despite the fact that $B = \alpha(Y, Z)$ is explicitly correlated with $S_1 = YZ$. Such behavior stems from the fact that the risk associated with the unboundedness of the underlying outweighs the risk associated with the bounded claim. This should be contrasted with the case of locally bounded markets, where a correlated claim often leads to a different optimizer for the primal problem.

4. THE INDIFFERENCE PRICE π

Consider an agent with utility u, initial endowment x, and investment possibilities given by $\mathcal{H}^W$ who seeks to sell a claim B. As pointed out in Section 1, the indifference price $\pi(B)$ for this claim is defined as the implicit solution to (1.1). In view of the results of Theorem 3.8, we now rephrase this definition in terms of the value function

$$U^W_B(x) := \sup_{k\in K^W_B} E[u(x + k - B)]. \tag{4.1}$$

Comparing this with (3.15), we see that we could alternatively denote (4.1) by U^W_{B-x}. In this section we prefer $U^W_B(x)$, since it better illustrates the different financial roles played by the initial endowment x and the claim B.

DEFINITION 4.1. Provided that the related maximization problems are well posed, the seller's indifference price $\pi(B)$ of the claim B is the implicit solution of the equation

$$U_0^W(x) = U_B^W(x + \pi(B)), \tag{4.2}$$

that is, $\pi(B)$ is the additional initial money that makes the optimal utility with the liability B equal to the optimal utility without B.

The next proposition lists the various properties of the indifference price *functional* π, defined on the set $\mathcal{B} \subseteq L^{\widehat{u}}$. Some results are new, in particular the regularity of the map and the description of the conjugate π^* and of the subdifferential $\partial\pi$. They are nice consequences of the choice of the natural Orlicz framework and the proofs are quite short and easy. The other items are extensions of well-established results to the present general setup (see, e.g., Becherer 2003 or the recent Owen and Zitkovich 2009, proposition 7.5 and the references therein). A recent reference book for the necessary notions from Convex Analysis is Borwein and Zhu (2005).

PROPOSITION 4.2. *Let $u : \mathbb{R} \to \mathbb{R}$ be an increasing and concave function satisfying $\lim_{x\to\infty} u(x) = -\infty$ and suppose that $U_0^W(x) < u(+\infty)$. Then the seller's indifference price $\pi : \mathcal{B} \to \mathbb{R}$ verifies the following properties:*

(1) π is a well-defined, convex, monotone nondecreasing functional and satisfies the cash additivity property: $\pi(B + c) = \pi(B) + c$, for any $B \in \mathcal{B}$, $c \in \mathbb{R}$.
(2) Regularity: π is norm continuous and subdifferentiable.
(3) Dual representation: π admits the representation

$$\pi(B) = \max_{Q\in\mathcal{M}_\Phi^W} (Q(B) - \alpha(Q)) \tag{4.3}$$

where the (minimal) penalty term $\alpha(Q)$ is given by

$$\alpha(Q) = x + \|Q^s\| + \inf_{\lambda>0} \left\{ \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda} \right\}.$$

As a consequence, the subdifferential $\partial\pi(B)$ of π at B is given by

$$\partial\pi(B) = \mathcal{Q}_B^W(x + \pi(B)), \tag{4.4}$$

where $\mathcal{Q}_B^W(x + \pi(B))$ is the set of minimizers of the dual problem associated with the right-hand side of (4.2).
(4) Bounds: π satisfies the bounds

$$\max_{Q\in\mathcal{Q}_0^W(x)} Q(B) \le \pi(B) \le \sup_{Q\in\mathcal{M}_\Phi^W} Q(B).$$

If $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$, the bounds above simplify to

$$E_{Q^*}[B] \le \pi(B) \le \sup_{Q\in M_\sigma\cap\mathcal{L}_\Phi} E_Q[B],$$

where the probability $Q^* \in M_\sigma \cap \mathcal{L}_\Phi$ is the unique dual minimizer in $\mathcal{Q}_0^W(x)$.

(5) *Volume asymptotics: For any $B \in \mathcal{B}$ we have*

$$\lim_{b \downarrow 0} \frac{\pi(bB)}{b} = \max_{Q \in \mathcal{Q}_0^W(x)} Q(B). \tag{4.5}$$

If B is in $M^{\widehat{u}}$,

$$\lim_{b \to +\infty} \frac{\pi(bB)}{b} = \sup_{Q \in \mathcal{M}_\Phi^W} Q(B). \tag{4.6}$$

If $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$, the two volume asymptotics above become

$$\lim_{b \downarrow 0} \frac{\pi(bB)}{b} = E_{Q^*}[B], \qquad \lim_{b \to +\infty} \frac{\pi(bB)}{b} = \sup_{Q \in M_\sigma \cap P_\Phi} E_Q[B].$$

(6) *Price of replicable claims: If $B \in \mathcal{B}$ is replicable in the sense that $B = c + \int_0^T H_t \, dS_t$ with $H \in \mathcal{H}^W$, but also $-H \in \mathcal{H}^W$, then $\pi(B) = c$.*

The proof of Proposition 4.2 is postponed to the end of this section. For now, let us comment on the financial significance of the volume asymptotics in item (5), which generalize known results in the topic. The limit in (4.5) leads to a "generalized pricing by marginal utility." Classic pricing by marginal utility can be recovered in the simpler case when W and B are in the Morse space $M^{\widehat{u}}$. In this case, there is only one $Q^* \in \mathcal{Q}_0^W(x)$, it is a probability measure and it is proportional to $u'(f_B - B)$, the marginal utility from the optimal primal solution (as given by relation (3.18)).

Regarding the second volume asymptotics (4.6) obtained when B is in $M^{\widehat{u}}$, suppose for this analysis that W is also in $M^{\widehat{u}}$. Then, this asymptotic behavior gives the "weak-super replication" price for B as defined and studied in Biagini and Frittelli (2004) for $W = 1$. This price is in general smaller than the super-replication price, as preferences are taken into account in the restriction of the class of pricing measures, from M_σ to $\mathcal{M}_\Phi^W$.

To better compare our results with the current literature, in the next corollary we specify the formula for π in the exponential utility case.

COROLLARY 4.3. *Let $u(x) = -e^{-\gamma x}$ and assume that $\mathcal{M}_\Phi^W \neq \emptyset$. If $B \in \mathcal{B}$ then:*

$$\pi_\gamma(B) = \max_{Q \in \mathcal{M}_\Phi^W} \left[Q(B) - \frac{1}{\gamma} \mathbb{H}(Q, P) \right], \tag{4.7}$$

where the penalty term is given by

(4.8)

$$\mathbb{H}(Q, P) := \gamma \| Q^s \| + H(Q^r | P) - U_0^W = \gamma \| Q^s \| + H(Q^r | P) - \min_{Q \in \mathcal{M}_\Phi^W} \{ \gamma \| Q^s \| + H(Q^r | P) \}.$$

Observe that for the exponential utility, the condition that $U_0^W(x) < u(+\infty)$ is equivalent to $\mathcal{M}_\Phi^W \neq \emptyset$. Apart from the presence of the singular term $\| Q^s \|$, the result in this corollary coincides with equation (5.6) of Delbaen et al. (2002).

In the next corollary we consider the risk measure induced by π:

COROLLARY 4.4. *Under the same hypotheses of Proposition 4.2, the seller's indifference price π defines a convex risk measure on $\mathcal{B}$, with the following representation:*

$$\rho(B) = \pi(-B) = \max_{Q \in \mathcal{M}_\Phi^W} \{ Q(-B) - \alpha(Q) \}. \tag{4.9}$$

If both the loss control W and the claim B are in $M^{\widehat{u}}$, then this risk measure has the Fatou property. In terms of π, this means

$$B_n \uparrow B \Rightarrow \pi(B_n) \uparrow \pi(B). \tag{4.10}$$

Proof. The first part is a consequence of Proposition 4.2 and the second part follows again from the fact that when W, B are in $M^{\widehat{u}}$ there is a version of the dual problem only with regular elements $Q \in \mathcal{M}_\Phi^W \cap L^1 = \mathbb{M}_\sigma \cap \mathcal{L}_\Phi$. Consequently, there is a representation $\rho(B) = \max_{Q \in \mathcal{M}_\Phi^W \cap L^1} \{Q(-B) - \alpha(Q)\}$ on the order continuous dual. But this implies the Fatou property (see, e.g., Biagini and Frittelli 2009, proposition 23). $\square$

4.1. Proof of Proposition 4.2

The next lemma establishes the properties of the set $\mathcal{B}$.

LEMMA 4.5. *The set $\mathcal{B}$ satisfies*

$$\mathcal{B} = \{B \in L^{\widehat{u}} \mid (-B) \in \mathrm{int}(\mathrm{Dom}(I_u))\} \tag{4.11}$$

and therefore has the properties:

1. *$\mathcal{B}$ is convex and open in $L^{\widehat{u}}$;*
2. *if $B_1 \in \mathcal{B}$ and $B_2 \le B_1$, then $B_2 \in \mathcal{B}$;*
3. *$\mathcal{B}$ contains $M^{\widehat{u}}$ (and thus L^∞);*
4. *for any given $B \in \mathcal{B}$ and $C \in M^{\widehat{u}}$, we have that $B + C \in \mathcal{B}$. In particular, $B + c \in \mathcal{B}$ for all constants $c \in \mathbb{R}$.*

Proof. The claim B satisfies (3.2) iff $-B^+ \in \mathrm{int}(\mathrm{Dom}(I_u))$. This is a consequence of lemma 30 in Biagini and Frittelli (2008), which in turn is based on the definition of the Luxemburg norm on $L^{\widehat{u}}$ and on a simple convexity argument as in Lemma 3.2. Since $B \in L^{\widehat{u}}$, B satisfies (3.2) iff $-B \in \mathrm{int}(\mathrm{Dom}(I_u))$.

Then, $\mathcal{B}$ is obviously open and convex (item 1) and item 2 is a consequence of the monotonicity of I_u. It is evident that $M^{\widehat{u}}$ is contained in $\mathcal{B}$, since $C \in M^{\widehat{u}}$ iff $E[\widehat{u}(kC)] < +\infty$ for all $k > 0$ (item 3). In order to prove item 4, fix $B \in \mathcal{B}$ and a convenient ϵ. For any C in $M^{\widehat{u}}$, set $r = \frac{\frac{\epsilon}{2}}{(1+\epsilon)(1+\frac{\epsilon}{2})}$. Then

$$E\left[u\left(-\left(1+\frac{\epsilon}{2}\right)(B+C)^+\right)\right] \ge \frac{1+\frac{\epsilon}{2}}{1+\epsilon} E[u(-(1+\epsilon)B^+)] + \frac{\epsilon/2}{1+\epsilon} E\left[u\left(-\frac{C^+}{r}\right)\right] > -\infty.$$

$\square$

Proof of Proposition 4.2. First observe that the assumption $U_0^W(x) < u(+\infty)$ always implies $\mathcal{M}_\Phi^W \neq \varnothing$, thanks to the dual formula in Theorem 3.8 with $B = 0$ (which then reduces to theorem 29 in Biagini and Frittelli 2008).

(1) We first need to show that the solution to the equation (4.2) exists and it is unique. Let $F(p) := U_B^W(x + p)$. By standard arguments it can be shown that $F : \mathbb{R} \to (-\infty, u(+\infty)]$ is concave and monotone nondecreasing, though not necessarily

strictly increasing. By monotone convergence,

$$\lim_{p\to+\infty} F(p) = u(+\infty). \tag{4.12}$$

We also have that $\lim_{p\to-\infty} F(p) = -\infty$, so that $F(p)$ is not constantly equal to $u(+\infty)$. Indeed, fix $Q \in \mathcal{M}_\Phi^W$ and take $\lambda > 0$ for which $E[\Phi(\lambda \frac{dQ^r}{dP})]$ is finite. As in the proof of Theorem 3.8, it follows from Fenchel's inequality and the definition of the set K_B^W that

$$F(p) = U_B^W(x+p) \le \lambda(x + p - Q(B) + \| Q^s \|) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right],$$

and then one obtains $\lim_{p\to-\infty} F(p) = -\infty$. The well-posedness of the definition of π is now straightforward. In fact, let p_L be the infimum of the set $\{p \in \mathbb{R} \mid F(p) = F(+\infty) = u(+\infty)\}$. From concavity, on $(-\infty, p_L)$ F is continuous and strictly monotone and thus a bijection onto the image $(-\infty, u(+\infty))$. Since $U_0^W(x) < u(+\infty)$, there always exists a unique p such that $F(p) = U_0^W(x)$, namely the indifference price $\pi(B)$.

Convexity and monotonicity are consequences of the definition (4.2), of the concavity and monotonicity of u. The cash-additivity property $\pi(B + c) = \pi(B) + c$ for any $c \in \mathbb{R}$, $B \in \mathcal{B}$, follows directly from the definition (4.2) and from the well-posedness of π.

(2) For this item, observe that π is a real valued, convex, monotone functional on the convex open subset $\mathcal{B}$ of the Banach lattice $L^{\widehat{u}}$. It then follows from item 2 of Lemma 4.5 that the extension $\widetilde{\pi}$ of π on $L^{\widehat{u}}$ with the value $+\infty$ on $L^{\widehat{u}} \backslash \mathcal{B}$ is still monotone, convex, and cash additive. Trivially, the interior of the proper domain of $\widetilde{\pi}$ coincides with $\mathcal{B}$. Therefore, norm continuity and subdifferentiability of $\widetilde{\pi}$ (and thus of π) on $\mathcal{B}$ follow from an extension of the classic Namioka-Klee theorem for convex monotone functionals (see Ruszczynski and Shapiro 2006, but also Biagini and Frittelli 2009 and Cheridito and Li 2009 in the context of Risk Measures). As a consequence, π admits a dual representation on $\mathcal{B}$ as

$$\pi(B) = \widetilde{\pi}(B) = \max_{Q \in (L^{\widehat{u}})^*_+,\, Q(\mathbf{1}_\Omega)=1} \{Q(B) - \pi^*(Q)\}, \tag{4.13}$$

where π^* is the convex conjugate of $\widetilde{\pi}$, that is $\pi^* : (L^{\widehat{u}})^* \to (-\infty, +\infty]$,

$$\pi^*(z) = \sup_{B' \in L^{\widehat{u}}} \{z(B') - \widetilde{\pi}(B')\} = \sup_{B \in \mathcal{B}} \{z(B) - \pi(B)\}.$$

The normalization condition $Q(\mathbf{1}_\Omega) = 1$ in (4.13) derives from the cash additivity property. It is a general result from Convex Analysis (and can be proved in a couple of lines) that there exists a dual representation of a (lower semicontinuous) convex functional ψ with the "max," i.e., $\psi(B) = \max_Q\{Q(B) - \psi^*(Q)\}$ if and only if ψ is subdifferentiable at B and the subdifferential set is given by those Q which attain the max, $\partial\psi(B) = \operatorname{argmax}\{Q(B) - \psi^*(Q)\}$. The subdifferential of π at B is therefore given by

$$\partial\pi(B) = \operatorname{argmax}\{Q(B) - \pi^*(Q)\}. \tag{4.14}$$

Note that, since $\pi(0) = 0$, π^* is nonnegative and thus it can be interpreted as a penalty function. The next item presents a characterization of π^* and therefore of $\partial\pi(B)$.

(3) A dual representation for π has just been obtained in (4.13). The current item is proved in two steps: first, we establish representation (4.3) with the penalty α; second, we prove that $\alpha = \pi^*$, that is α is the *minimal penalty* function, which together with (4.14) gives (4.4) and completes the proof.

Step 1. From the definition of $\pi(B)$ and from the dual formula (3.17)

$$\begin{aligned} U_0^W(x) &= U_B^W(x+\pi(B)) \\ &= \min_{\lambda>0,\, Q\in\mathcal{M}_\Phi^W} \left\{ \lambda Q(-B+x+\pi(B)) + \lambda \|Q^s\| + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right]\right\}. \end{aligned}$$

Necessarily then

$$\pi(B) \geq Q(B) - \left[x + \|Q^s\| + \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda}\right] \text{ for all } \lambda > 0,\ Q \in \mathcal{M}_\Phi^W$$

and equality holds for the optimal λ^* and any $Q^* \in \mathcal{Q}_B^W(x+\pi(B))$. Fixing $Q \in \mathcal{M}_\Phi^W$ and taking first the supremum over $\lambda > 0$, we get

$$\pi(B) \geq Q(B) - \inf_{\lambda>0}\left[x + \|Q^s\| + \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda}\right].$$

Taking then the supremum over Q we finally obtain

$$\pi(B) = \max_{Q\in\mathcal{M}_\Phi^W}\left\{ Q(B) - \inf_{\lambda>0}\left[x + \|Q^s\| + \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda}\right]\right\}$$

where equality holds for λ^*, $Q^* \in \mathcal{Q}_B^W(x+\pi(B))$. Observe that the following extension, still denoted by α,

$$\alpha(Q) = \begin{cases} \inf_{\lambda>0}\left[x + \|Q^s\| + \dfrac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda}\right] & \text{when } Q \in \mathcal{M}_\Phi^W \\ +\infty & \text{otherwise} \end{cases}$$

is $[0,+\infty]$-valued, satisfies $\inf_{Q\in(L^{\widehat{u}})^*}\alpha(Q) = 0$ and therefore it is a *grounded* penalty function. Clearly

$$\pi(B) = \max_{Q\in(L^{\widehat{u}})^*_+}\{Q(B) - \alpha(Q)\}$$

and

$$\operatorname{argmax}\{Q(B) - \alpha(Q)\} = \mathcal{Q}_B^W(x+\pi(B)) \tag{4.15}$$

as sets. In particular, when $B = 0$

$$\pi(0) = 0 \quad \text{and} \quad \operatorname{argmax}\{-\alpha(Q)\} = \operatorname{argmin}\{\alpha(Q)\} = \mathcal{Q}_0^W(x). \tag{4.16}$$

Step 2. As α provides another penalty function, a basic result in convex duality ensures that $\pi^* = \alpha^{**}$, i.e., π^* is the convex, $\sigma((L^{\widehat{u}})^*, L^{\widehat{u}})$ —lower semicontinuous hull of α. We want to show that $\pi^* = \alpha$. To this end, we prove that α is *already* convex and lower semicontinuous.

(a) α is convex: Let $Q(y) = yQ_1 + (1-y)Q_2$ be the convex combination of any couple of elements in $\mathcal{M}_\Phi^W$ (if the Q_i are not in $\mathcal{M}_\Phi^W$ there is nothing to prove). Given any $\lambda_1, \lambda_2 > 0$, define $\lambda(y) = \frac{1}{(1-y)\frac{1}{\lambda_2} + y\frac{1}{\lambda_1}}$, so that $\frac{1}{\lambda(y)} = (1-y)\frac{1}{\lambda_2} + y\frac{1}{\lambda_1}$. Then

$$\begin{aligned}
\alpha(Q(y)) &\leq \left[x + \|Q^s(y)\| + \frac{E\left[\Phi\left(\lambda(y)\frac{dQ^r(y)}{dP}\right)\right] - U_0^W(x)}{\lambda(y)} \right] \\
&\leq y\left[x + \|Q_1^s\| + \frac{E\left[\Phi\left(\lambda_1\frac{dQ_1^r}{dP}\right)\right] - U_0^W(x)}{\lambda_1} \right] \\
&\quad + (1-y)\left[x + \|Q_2^s\| + \frac{E\left[\Phi\left(\lambda_2\frac{dQ_2^r}{dP}\right)\right] - U_0^W(x)}{\lambda_2} \right]
\end{aligned}$$

where the inequalities follow from the convexity of the norm and of the function $(z, k) \to z\Phi(k/z)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, as already pointed out. Taking the infimum over λ_1 and λ_2 we get: $\alpha(Q(y)) \leq y\alpha(Q_1) + (1-y)\alpha(Q_2)$.

(b) α is lower semicontinuous: Since α is a convex map on a Banach space, weak lower semicontinuity is equivalent to norm lower semicontinuity. Suppose then that Q_k is a sequence converging to Q with respect to the Orlicz norm. We must prove that

$$\alpha(Q) \leq \liminf_k \alpha(Q_k) := L.$$

We can assume $L = \liminf_k \alpha(Q_k) < +\infty$, otherwise there is nothing to prove. Now, it is not difficult to see that

$$Q_k \overset{\|\cdot\|}{\to} Q \text{ iff } Q_k^r \overset{\|\cdot\|}{\to} Q^r,\ Q_k^s \overset{\|\cdot\|}{\to} Q^s \tag{4.17}$$

so that $Q_k^r \to Q^r$ in $L^{\widehat{\Phi}}$ and henceforth in L^1. We can extract a subsequence, still denoted by Q_k to simplify notation, such that $\alpha(Q_k) \to L$ and $Q_k^r \to Q^r$ *a.s.* So these Q_k are (definitely) in $\mathcal{M}_\Phi^W$, which is closed and therefore the limit $Q \in \mathcal{M}_\Phi^W$.

From the definition of α we deduce that for all $k \in \mathbb{N}_+$ there exists $\lambda_k > 0$ such that

$$\alpha(Q_k) \leq x + \|Q_k^s\| + \frac{E\left[\Phi\left(\lambda_k\frac{dQ_k^r}{dP}\right)\right] - U_0^W(x)}{\lambda_k} \leq \alpha(Q_k) + \frac{1}{k}.$$

The next arguments rely on a couple of applications of Fatou's Lemma to (a subsequence of) the sequence $\left(\frac{\Phi(\lambda_k\frac{dQ_k^r}{dP}) - U_0^W(x)}{\lambda_k}\right)_k$. Fatou's Lemma is enabled here by the condition $U_0^W(x) < u(+\infty)$ and by the convergence of the regular parts $(\frac{dQ_k^r}{dP})_k$. In fact, one can always find an $\widetilde{x}$ such that $u(\widetilde{x}) = U_0^W(x)$ and then the Fenchel inequality $\Phi(y) - u(\widetilde{x}) + y\widetilde{x} \geq 0$ gives the required control from below

$$\frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right) - U_0^W(x)}{\lambda_k} + \frac{dQ_k^r}{dP}\widetilde{x} \geq 0. \tag{4.18}$$

The sequence $(\lambda_k)_k$ cannot tend to $+\infty$. In fact, if $\lambda_k \to +\infty$, then a.s. we would have (remember that Φ is bounded below)

$$\begin{aligned}
\liminf_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right) - U_0^W(x)}{\lambda_k} &= \liminf_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right)}{\lambda_k} \\
&\geq \lim_k \frac{(\min_y \Phi(y))}{\lambda_k} \mathbf{1}_{\{\frac{dQ_k^r}{dP} \wedge \frac{dQ^r}{dP} = 0\}} + \lim_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right)}{\lambda_k} \mathbf{1}_{\{\frac{dQ_k^r}{dP} \wedge \frac{dQ^r}{dP} > 0\}} \\
&= \lim_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right)}{\lambda_k \frac{dQ_k^r}{dP}} \frac{dQ_k^r}{dP} \mathbf{1}_{\{\frac{dQ_k^r}{dP} > 0\}} \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}}.
\end{aligned} \tag{4.19}$$

Since $\mathbf{1}_{\{\frac{dQ_k^r}{dP} > 0\}} \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}} \to \mathbf{1}_{\{\frac{dQ^r}{dP} > 0\}}$ a.s. and, as already checked, $\lim_{y \to +\infty} \frac{\Phi(y)}{y} = +\infty$ the limit in (4.19) is in fact $+\infty$ on the set $\{\frac{dQ^r}{dP} > 0\}$ which has positive probability as $Q \in \mathcal{M}_\Phi^W$. But then

$$\begin{aligned}
L = \lim_k \left\{\alpha(Q_k) + \frac{1}{k}\right\} &\geq \lim_k \left\{ x + \| Q_k^s \| + E\left[\frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right) - U_0^W(x)}{\lambda_k}\right]\right\} \\
&\geq x + \| Q^s \| + E\left[\liminf_k \frac{\Phi\left(\lambda_k \frac{dQ_k^r}{dP}\right) - U_0^W(x)}{\lambda_k}\right] = +\infty,
\end{aligned}$$

where in the inequality we apply (4.17) and Fatou's Lemma.

Therefore there exists some compact subset of $\mathbb{R}_+$ that contains λ_k for infinitely many k's, so that we can extract a subsequence $\lambda_{k_n} \to \lambda^*$. The inequality (4.18) ensures that λ^* must be strictly positive. Otherwise, if $\lambda^* = 0$, the numerator of the fraction there tends to $\Phi(0) - U_0^W(x) = u(+\infty) - U_0^W(x) > 0$ and globally the limit random variable would be $+\infty$. Finally,

$$\begin{aligned}
\alpha(Q) &\leq x + \| Q^s \| + \frac{E\left[\Phi\left(\lambda^* \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda^*} \\
&\leq x + \liminf_n \left\{ \| Q_{k_n}^s \| + \frac{E\left[\Phi\left(\lambda_{k_n} \frac{dQ_{k_n}^r}{dP}\right)\right] - U_0^W(x)}{\lambda_{k_n}} \right\} = L.
\end{aligned}$$

Hence, $\alpha = \pi^*$ and the identity $\partial\pi(B) = \mathcal{Q}_B^W(x + \pi(B))$ in (4.4) follows from (4.14) and (4.15).

(4) The bounds

$$\sup_{Q\in\mathcal{Q}_0^W(x)} Q(B) \leq \pi(B) \leq \sup_{Q\in\mathcal{M}_\Phi^W} Q(B) \tag{4.20}$$

are easily proved, since the first inequality follows from the fact that when $Q \in \mathcal{Q}_0^W(x)$, the penalty $\alpha(Q) = 0$ (see (4.16)) and the second inequality holds because α is a penalty, i.e., $\alpha(Q) \geq 0$. The first supremum is in fact a maximum, which is a consequence of the "Max Formula" as better explained in item (5) below. The case $W, B \in \widehat{M^u}$ is immediate from (4.20) and from the special form of the dual as stated in Corollary 3.9.

(5) Let $\pi'(C, B)$ indicate the directional derivative of π at C along the direction B, i.e., $\pi'(C, B) = \lim_{b\downarrow 0} \frac{\pi(C+bB)-\pi(C)}{b}$. The so-called Max Formula (Borwein and Zhu 2005, theorem 4.2.7) states that given a convex function π and a continuity point C, then

$$\pi'(C, B) = \max_{Q\in\partial\pi(C)} Q(B).$$

So the first volume asymptotic becomes a trivial application of the Max Formula with $C = 0$, since $bB \in \mathcal{B}$ if $b \leq 1 + \epsilon$ and

$$\lim_{b\downarrow 0} \frac{\pi(bB)}{b} = \pi'(0, B) = \max_{Q\in\mathcal{Q}_0^W(x)} Q(B),$$

because $\pi(0) = 0$ and $\mathcal{Q}_0^W(x) = \partial\pi(0)$.

For the second volume asymptotic, when $B \in \widehat{M^u}$ then $bB \in \mathcal{B}$ for all $b \in \mathbb{R}$. So, $\pi(bB)$ is well defined and for all $b > 0$ we have that $\pi(bB) \leq \sup_{Q\in\mathcal{M}_\Phi^W} Q(bB)$. Therefore

$$\limsup_{b\to+\infty} \frac{\pi(bB)}{b} \leq \sup_{Q\in\mathcal{M}_\Phi^W} Q(B).$$

If we fix $Q \in \mathcal{M}_\Phi^W$, the penalty $\alpha(Q)$ is finite and $\frac{\pi(bB)}{b} \geq Q(B) - \frac{\alpha(Q)}{b}$ for all $b > 0$ so that $\liminf_{b\to+\infty} \frac{\pi(bB)}{b} \geq Q(B)$ for all $Q \in \mathcal{M}_\Phi^W$ and $\lim_{b\to+\infty} \frac{\pi(bB)}{b} = \sup_{Q\in\mathcal{M}_\Phi^W} Q(B)$.

Finally, the case $W, B \in \widehat{M^u}$ follows from the asymptotics just proved and Corollary 3.9.

(6) If B and $-B$ are replicable with admissible strategies, then $Q(B) = c$ for all $Q \in \mathcal{M}_\Phi^W$, hence in particular for the "zero penalty functionals" $Q \in \mathcal{Q}_0^W(x)$. Therefore $\pi(B) = \max_Q\{Q(B) - \alpha(Q)\} = c$. □

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