Solutions to Assignment 1

1. The Tarski-Vaught Test

We prove the TV test by induction on formulas:

i. For atomic formulas, the statement is automatic by the definition of substructure.

ii. Connectives are straightforward.

iii. Suppose $\varphi(x_1 \dots x_n)$ is $\exists y\, \psi(y, x_1 \dots x_n)$ and $a_1 \dots a_n \in N$. If $\mathcal{M} \models \varphi(a_1 \dots a_n)$ then there is $b \in N$ s.t. $\mathcal{M} \models \psi(b, a_1 \dots a_n)$ and so by induction $\mathcal{N} \models \psi(b, a_1 \dots a_n)$ which proves one direction.

The other direction is the assumption in the TV test: If $\mathcal{M} \models \varphi(a_1 \dots a_n)$ then there is a $\underline{b \in M}$ s.t. $\mathcal{N} \models \psi(b, a_1 \dots a_n)$. By induction $\mathcal{M} \models \psi(b, a_1 \dots a_n)$ so $\mathcal{M} \models \varphi(a_1 \dots a_n)$.

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2. Łoś theorem for classical logic.

Assume we are given $\mathcal{M}_i$, $i \in I$ all $\mathcal{L}$-structures and $\mathcal{U}$, an u.f. on I. We prove this theorem by induction on formulas. Let $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$

i. Atomic formulas are automatic by definition of the ultraproduct.

ii. For connectives, we treat $\varphi \wedge \psi$ and $\neg\varphi$.

$\varphi \wedge \psi$: Suppose $a^1 \cdots a^n \in M$

$$\mathcal{M} \models (\varphi \wedge \psi)(a^1 \cdots a^n) \text{ iff } \mathcal{M} \models \varphi(a^1 \cdots a^n) \text{ and } \mathcal{M} \models \psi(a^1 \cdots a^n)$$

$$\text{iff } \{i \in I : \mathcal{M}_i \models \varphi(a_i^1 \cdots a_i^n)\} \in \mathcal{U}$$

$$\text{and } \{i \in I : \mathcal{M}_i \models \psi(a_i^1 \cdots a_i^n)\} \in \mathcal{U}.$$

(by induction).

$$\text{iff } \{i \in I : \mathcal{M}_i \models (\varphi \wedge \psi)(a_i^1 \cdots a_i^n)\} \in \mathcal{U}.$$

(by the fact the u.f. are closed under intersection and supersets).

$\neg\varphi$: $\mathcal{M} \models \neg\varphi(a^1 \cdots a^n)$ iff $\mathcal{M} \not\models \varphi(a^1 \cdots a^n)$

iff $\{i \in I : \mathcal{M}_i \models \varphi(a_i^1 \cdots a_i^n)\} \notin \mathcal{U}$ (by induction),

iff $\{i \in I : \mathcal{M}_i \models \neg\varphi(a_i^1 \cdots a_i^n)\} \in \mathcal{U}$,

(since u.f. are closed under complements).

iii. Suppose $\varphi := \exists y\, \psi(y, x_1 \cdots x_n)$

If $\mathcal{M} \models \exists y\, \psi(y, a^1 \cdots a^n)$ then there is some $b \in M$ s.t. $\mathcal{M} \models \psi(b, a^1 \cdots a^n)$.

By induction then $\{i \in I : \mathcal{M}_i \models \psi(b_i, a_i^1 \cdots a_i^n)\} \in \mathcal{U}$

and this set is certainly contained in

$\{i \in I : \mathcal{M}_i \models \exists y\, \psi(y, a_i^1 \cdots a_i^n)\}$ so it is in $\mathcal{U}$.

In the other direction, assume $Y = \{i \in I : \mathcal{M}_i \models \varphi(a_i^1 \cdots a_i^n)\} \in \mathcal{U}$.

If $i \in Y$ then pick $b_i \in M_i$ s.t. $\mathcal{M}_i \models \psi(b_i, a_i^1 \cdots a_i^n)$
If $i \notin Y$ then let b_i be any element of M_i.

We then have $\mathcal{M} \models \psi(b, a^1 \cdots a^n)$ by induction where $b = (b_i : i \in I)$ and so $\mathcal{M} \models \varphi(a^1 \cdots a^n)$.

3. Let $M = \bigcup_{i<\alpha} M_i$ and define an $\mathcal{L}$-structure

on M as follows: If f is an n-ary function symbol and $a_1 \cdots a_n \in M$ then choose M_i so that $a_1 \cdots a_n \in M_i$ and let $f^{M}(a_1 \cdots a_n) = f^{M_i}(a_1 \cdots a_n)$. This is well-defined since if $a_1 \cdots a_n \in M_j$ for some j then either $M_i \prec M_j$ or $M_j \prec M_i$ and so at least M_i and M_j agree on their definition of function symbols.

Relation symbols are similar. In this way we have shown that $\mathcal{M}_i \subseteq \mathcal{M}$ and so there is agreement on atomic formulas. Again, in showing $\mathcal{M}_i \prec \mathcal{M}$, connectives do not cause a problem when doing this by induction.

If we have $\mathcal{M} \models \exists y\, \varphi(y, a_1 \ldots a_n)$ for $a_1 \ldots a_n \in M_i$ then $\mathcal{M} \models \varphi(b, a_1 \ldots a_n)$ for some $b \in M_j$ and we can assume $j \geq i$. By induction $\mathcal{M}_j \models \varphi(b, a_1 \ldots a_n)$ so

$\mathcal{M}_j \models \exists y\, \varphi(y, a_1 \ldots a_n)$ and since $\mathcal{M}_i \preceq \mathcal{M}_j$ we get

$\mathcal{M}_i \models \exists y\, \varphi(y, a_1 \ldots a_n)$. If on the other hand $\mathcal{M}_i \models \exists y\, \varphi(y, a_1 \ldots$

$\mathcal{M}_i \models \exists y\, \varphi(y, a_1 \ldots a_n)$ then for some $b \in M_i$,

$\mathcal{M}_i \models \varphi(b, a_1 \ldots a_n)$ and so by induction $\mathcal{M} \models \varphi(b, a_1 \ldots a_n)$ from which we conclude that $\mathcal{M} \models \exists y\, \varphi(y, a_1 \ldots a_n)$.

$\boxtimes$

4. By scaling the r_i's if necessary, we can assume $r_i \in [0,1]$ for all $i \in I$.

We now define a sequence of integers k_j for all $j \in \mathbb{N}$ s.t.

① $0 \leq k_j < 2^j$ and.

② $\{ i \in I : r_i \in [\frac{k_j}{2^j}, \frac{k_j+1}{2^j}) \} \in \mathcal{U}$.

We should cover off a silly case right away:

~~If $r_i = \frac{M}{2^n}$ for some M, n~~

If there is an M, n s.t. $r_i = \frac{M}{2^n}$ for almost all $i \in I$ then $\lim_{\mathcal{U}} r_i = \frac{M}{2^n}$. Otherwise, the choice of k_j is unique for each j and the sequence.

$\frac{k_j}{2^j}$ is convergent so $\lim_{\mathcal{U}} r_i = \lim_{j \to \infty} \frac{k_j}{2^j}$.

$\blacksquare$

5. We know that the map $\Delta : X \to \prod_{\mathcal{U}} X$ sending x to the constant tuple $(x : i \in I)$ is an isometry. We just need to see that it is onto.

So pick $(x_i : i \in I) \in \prod_I X$. We want to choose $x \in X$ s.t. for every $\varepsilon > 0$

$$\{ i \in I : d(x, x_i) < \varepsilon \} \in \mathcal{U}.$$

Since (X, d) is compact, for each n we can choose finitely many points $x_1^n \dots x_{k_n}^n$ s.t. the $1/n$-balls centered at x_j^n cover X.

Now define a sequence $y_n = x_j^n$ for some j s.t.

$$\{ i \in I : d(x_i, x_j^n) < \tfrac{1}{n} \} \in \mathcal{U}.$$

Claim: $(y_n : n \in \mathbb{N})$ is a Cauchy sequence.

Pf/ Choose $\varepsilon > 0$ and m s.t. $\frac{1}{m} < \frac{\varepsilon}{2}$. If $n_1, n_2 > m$ then pick $i \in I$ s.t. $d(x_i, y_{n_1}) < \frac{1}{n_1}$ and $d(x_i, y_{n_2}) < \frac{1}{n_2}$. Then

$$d(y_{n_1}, y_{n_2}) < \frac{1}{n_1} + \frac{1}{n_2} < \frac{1}{m} + \frac{1}{m} < \varepsilon.$$

Let x = the limit of the y_n's and it is easy to check x works.