Applications of Chern-Simons Theory in knot theory

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Abstract

After an introduction to knot invariants and Chern-Simons gauge theory (CSGT), where we follow the exposition of J. Baez & J. Muniain[2], we note that vacuum expectation value (vev) of Wilson loops in CSGT gives a knot invariant for each representation for each group. Assuming some facts from conformal field theory (CFT) and closely following Witten[1], we calculate vev in CSGT using surgery methods for $G = U(1)$, $SU(2)$ (with the fundamental representation). The resulting knot invariants are the linking number and the Jones polynomial, respectively.

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1 Background Materials

1.1 Some Knot theory

A *knot* is a manifold in $\mathbb{R}^3$ diffeomorphic to S^1

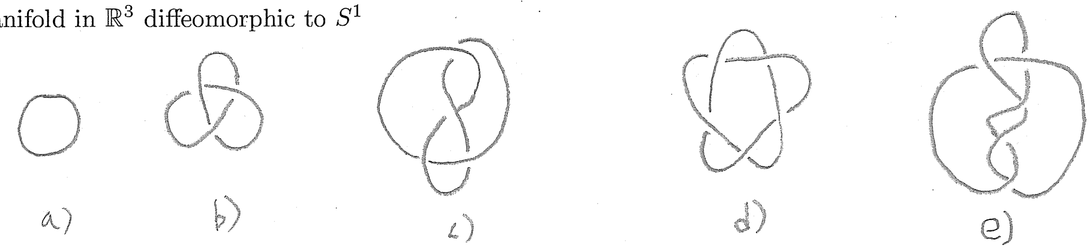


Fig. 1: Examples of knots, labelled by their minimal crossing number and a subscript index. a) unknot b) trefoil knot (3_1) c) figure-eight knot (4_1) d) 5_1 e) 5_2

A *link* is a manifold in $\mathbb{R}^3$ diffeomorphic to copies of S^1. Each piece diffeomorphic to S^1 is called a *link component.*

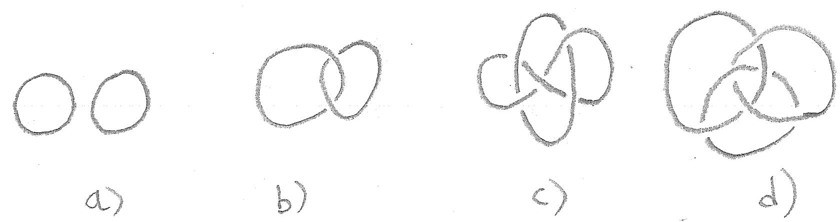


Fig. 2: Examples of links. a) unlink b) Hopf link c) Whitehead link d) Borromean Rings

If we make a trefoil knot with a piece of string, we find that it is impossible to deform it into an unknot without cutting it open. This motivates a notion of equivalence, between knots and links which can be deformed into one another.

We say a link L is *ambient equivalent* to L' if L is mapped to L' under an orientation-preserving diffeomorphism of $\mathbb{R}^3$. The orientation-preserving condition says we do not always identify a knot with its mirror image. One of the goals in knot theory is to provide a complete classification of knots and links, up to ambient equivalence.

It is not surprising that, just as the examples above, all knots can be represented by a knot diagram drawn in 2D. They are graphs for which each node has degree 4, representing a crossing between two segments of knots, with one segment running over another. It also turns out that two links are ambient equivalent if and only if their diagrams differ from one another through a sequence of *Reidemeister moves*:

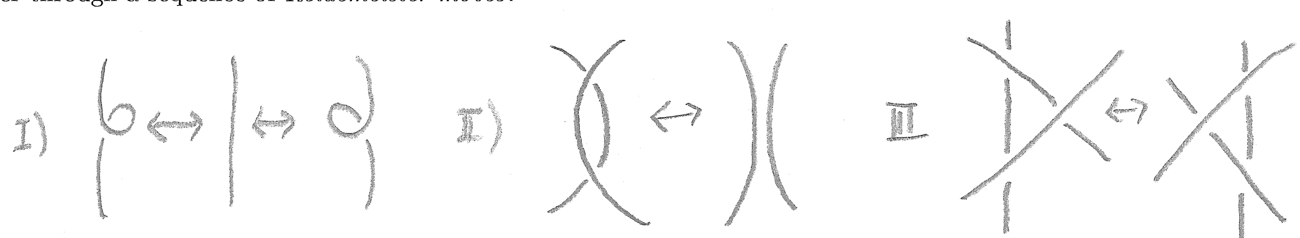


Fig. 3: The three Reidemeister moves

A *knot invariant* is a quantity $Inv(D(L))$ associated with each link diagram $D(L)$ such that $Inv(D(L))$ is unchanged under the Reidemeister moves (so we can simplify the notation and write $Inv(L)$ instead). If $Inv(K_1) \neq Inv(K_2)$ then the knots K_1, K_2 are not ambient equivalent. We will see two examples of knot invariants: the *linking number* and the *Jones polynomial.*

1.1.1 Linking Number

For each given llink diagram, we may give an arrow running on each link component. Such link diagrams are said to be *oriented*.

Fig. 4: Oriented links

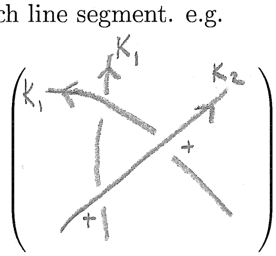
Each crossing c can be classified into two types, and we can associate a sign for each type:

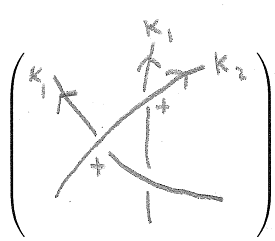
Fig. 5: positive(+) crossing negative(−) crossing

Let $\{K_i : i \in I\}$ be a link, where each K_i is a knot. Let C_{ij} be the set of crossings between K_i and K_j. Then the following is a knot invariant for $i \neq j$:

$$L(K_i, K_j) = \frac{1}{2} \sum_{c \in C_{ij}} sign(c) \tag{1.1}$$

to check, compute the change in L_{ij} under all Reidemeister moves with all possible assignments of orientation and membership to K_i, K_j, K_k for each line segment. e.g.

$L(K_1, K_2)$ 

$= L(K_1, K_2)$ 

(Notice we only sum on crossings between different links)

Inequivalent links can have the same linking number.

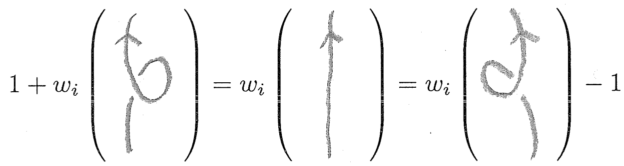
Framing & self-intersection number

We are tempted to compute the following number, which includes the contribution from C_{ii}:

$$\begin{aligned} w(K_1, \ldots, K_n) &:= \sum_{c \in all\ crossings} sign(c) = \sum_i \sum_{c \in C_{ii}} sign(c) + \sum_{i<j} \sum_{c \in C_{ij}} sign(c) \\ &= \sum_i w_i + 2\sum_{i<j} L(K_i, K_j) = \sum_i w_i + \sum_{i \neq j} L(K_i, K_j) \end{aligned} \tag{1.2}$$

where we have defined $w_i := \sum_{c \in C_{ii}} sign(c)$, the *writhe* or the *self-linking number* of K_i.

w_i is not invariant under type I moves.



To make (1.2) an invariant on each equivalence class of links, we refine our definition of equivalence class of links by introducing a "framing" for each link component. With this definition, two links can be inequivalent even if they have the same diagram with different framings.

A *framing* on a link component K_i is a smooth, non-vanishing vector field $\vec{f_i}$ on K_i such that at each point p of K_i, $\vec{f_i}$ is orthogonal to the tangent vector of K_i at p. The tips of $\vec{f_i}$ forms another link $\tilde{K_i}$, thought to be infinitesimally close to K_i. We may imagine a framing as a small, everywhere non-zero perturbation of K_i, and imagine a framed link as a 2-dimensional, ribbon-like object diffeomorphic to $S^1 \times [0,1]$, instead of a 1-dimensional object.

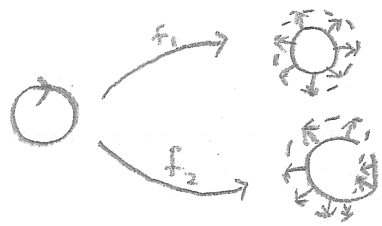


Fig. 6: Different framings for the unknot

A diffeomorphism h of $\mathbb{R}^3$ induces a map from the set of framings of K_i to the set of framings of $h(K_i)$. When we picture the framing as $\tilde{K_i}$, the framing induced on $h(K_i)$ is then $h(\tilde{K_i})$.

Two framed links $\{(K_i, \tilde{K_i}) : i = 1 \dots n\}$, $\{(K'_i, \tilde{K'_i}) : i = 1 \dots n\}$ are said to be *equivalent* if $\exists$ orientation preserving diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\{(K'_i, \tilde{K'_i}) : i = 1 \dots n\} = \{(h(K_i), h(\tilde{K_i})) : i = 1 \dots n\}$$

Given a oriented link diagram, we can make a particular choice of framing called the *blackboard framing*, by choosing f_i to always point to the left to the direction on the link.

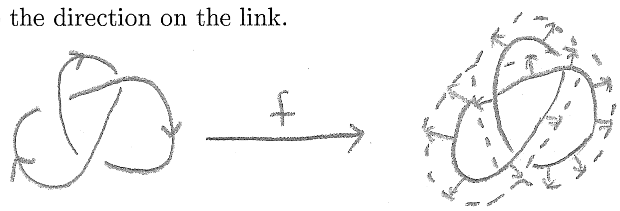


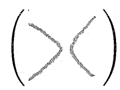
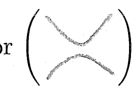
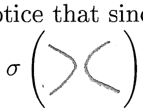
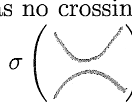
Fig. 7: Blackboard framing of the trefoil

Now one can check w_i is just $L(K_i, \tilde{K_i})$ with the blackboard framing. Since $L(K_i, \tilde{K_j}) = L(K_i, K_j)$ for $i \neq j$, we see that $w(K_1, \dots, K_n) = \sum_{i,j} L(K_i, \tilde{K_j})$. Equivalence of framed links implies ambient equivalence of links $K_i, \tilde{K_i} : i = 1 \dots n$, which implies invariance of $L(K_i, \tilde{K_j})$ for any i, j. So $w(K_1, \dots, K_n)$ is an invariant for equivalent framed links. $w(K_1, \dots, K_n)$ is not a link invariant because w_i is not a link invariant, which is because the blackboard framing on different diagrams may give different framings.

1.1.2 Jones Polynomial

To construct the Jones Polynomial, we first construct the *Kauffman bracket* for an oriented link diagram.

Kauffman bracket

A state assignment s for a diagram K is an assignment for each crossing c_i , a state $s_i =$  or . $K(s)$ is the diagram in which each c_i is replaced with s_i. Notice that since $K(s)$ has no crossing, it is just a collection of $m(s)$ circles $K(s) = \coprod_{j=1}^{m(s)} O$, where O denotes a circle. Let σ  $= +1$ and σ  $= -1$. The Kauffman bracket $< K > \in \mathbb{Z}[A]$ is defined by the following conditions:

1.
$$< K >= \sum_s [< K(s) > \prod_{c_i} A^{\sigma(s_i)}] \tag{1.3}$$

2.
$$< K \coprod O >= -(A^2 + A^{-2}) < K >< O > \tag{1.4}$$

3.
$$< O >= 1 \tag{1.5}$$

2. and 3. implies $< K(s) >= [-(A^2 + A^{-2})]^{m(s)-1}$. One can also check that the three conditions imply

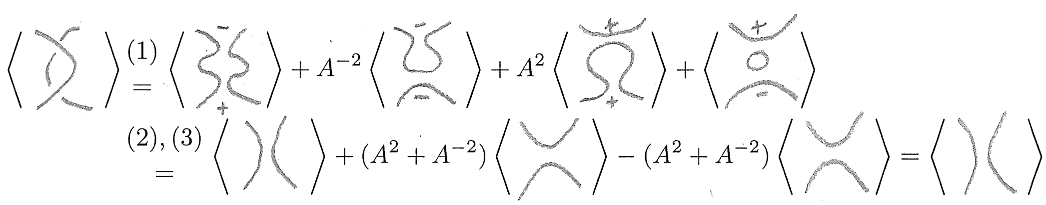
$$< K_1 \coprod K_2 >= -(A^2 + A^{-2}) < K_1 >< K_2 > \tag{1.6}$$

as a generalization of (2).

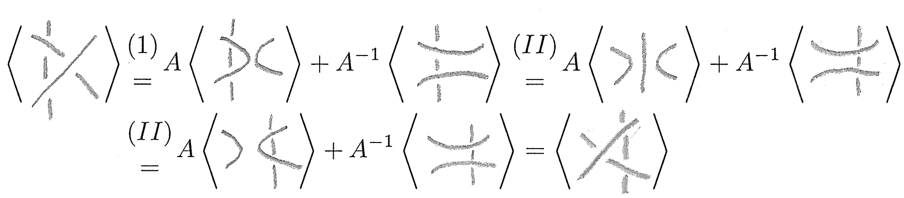
Claim. $< K >$ is invariant under type II and type III moves. Under type I moves, $< K >$ changes by a factor of $-A^3$

Proof. (Drawing a part of a diagram means all the rest of the diagram is the same for every term in the equation),

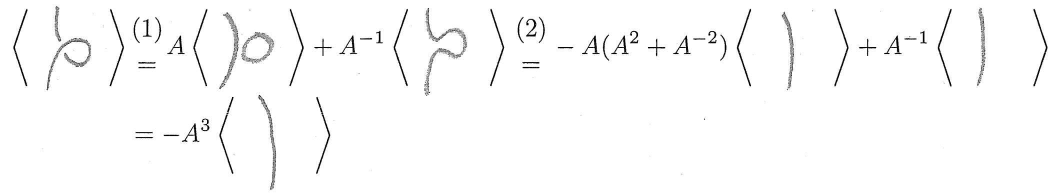
(II):



(III):



(I):



□

so $< K >$ is not a knot invariant. However, we know that $w(K)$ is also invariant under type II and III moves and changes by ± 1 under type I moves, so we can construct the quantity $V(K) := (-A^3)^{-w(K)} < K >$ which is invariant under all three moves. $V(K)$, with the change of variable $t^{\frac{1}{2}} = A^{-2}$ is called the *Jones polynomial.*

One can check $V(K)$ satisfies the following:

1. *Normalization:*

$$V(O) = 1 \tag{1.7}$$

2. *Skein relation:*

$$-t^{-1}V(L_+) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0) + tV(L_-) = 0 \tag{1.8}$$

where $L_+ =$  , $L_0 =$  , $L_- =$ 

Moreover, from (1.6),

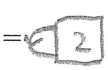
$$V(K_1 \coprod K_2) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})V(K_1)V(K_2) \tag{1.9}$$

Knowing that $V(K)$ is a knot invariant, (1.7) and (1.8) allow us to compute $V(K)$. This can be seen by induction on the number of crossings. If we were free to change any $L_\pm$ to $L_\mp$, the knot will eventually untie and become an unknot. The skein relation allows us to express $V(L_\pm)$ in terms of $V(L_\mp)$ and $V(L_0)$, the latter of which has one less crossing and is computable by the induction hypothesis. If there are no crossings, K is a collection of m circles, $V(K)$ can be computed by induction on m and considering diagrams such as

$-t^{-1}V($  $) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V($  $) + tV($  $) = 0$

In particular, this means (1.9) is just a rather indirect consequence of (1.7) and (1.8).

Connected sum of knots

If $K_1 =$  , $K_2 =$ , where the boxes may be any diagram, then the connected sum $K_1 + K_2 :=$ 

Using (1.8),(1.9),

$-t^{-1}V($  $) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V($  $) + tV($  $) = 0$

$$(-t^{-1} + t)V(K_1 + K_2) - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(t^{\frac{1}{2}} + t^{-\frac{1}{2}})V(K_1)V(K_2) = 0$$

so

$$V(K_1 + K_2) = V(K_1)V(K_2) \tag{1.10}$$

1.2 Chern-Simons Theory

1.2.1 Connection, curvature and Chern Form

Let G be a compact simple Lie group, let M^4 be a 4-manifold with G-bundle E, which is constructed as follows. Let $\{U_\alpha : \alpha \in I\}$ be charts covering M^4, each equipped with the trivial bundle $E_\alpha = U_\alpha \times V$, where V is a vector space on which G acts. Then we identify $E_\alpha(p) \sim g_{\alpha\beta}(p)E_\beta(p)$ for some $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Let A be a connection on E, so A is an $End(E)$-valued 1-form. Locally (i.e. in one chart U_α) $A_\alpha = A_{\alpha i}dx^i$. A_α, A_β from different charts differ by a gauge transformation[1] $A_\alpha = \widehat{g}_{\alpha\beta}(p)A_\beta := g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}$. Define curvature $F := dA + A \wedge A$, where $dA = \partial_i A_j dx^i \wedge dx^j$. F is well defined because it does not depend on gauge. Let there be $End(E)$-valued p-form $\omega = \omega_I dx^I$ and $End(E)$-valued q-form $\nu = \nu_J dx^J$, define $[\omega, \nu] := \omega \wedge \nu - (-1)^{pq}\nu \wedge \omega$, where $\omega \wedge \nu = (\omega_I \circ \nu_J)dx^I \wedge dx^J$. Define $d_A\omega := d\omega + [A, \omega]$. Then

$$\begin{aligned} d_AF &= d_A(dA + A \wedge A) \\ &= d^2A + dA \wedge A - A \wedge dA + [A, dA + A \wedge A] \\ &= dA \wedge A - A \wedge dA + A \wedge dA - dA \wedge A + [A, A \wedge A] \\ &= 0 \end{aligned}$$

if we perturb $A \to A + \delta A$, then to first order in δA,

$$\begin{aligned} F + \delta F &= d(A + \delta A) + (A + \delta A) \wedge (A + \delta A) \\ \delta F &= d(\delta A) + \delta A \wedge A + A \wedge \delta A \\ &= d_A \delta A \end{aligned} \tag{1.11}$$

Let $tr(\omega) = tr(\omega_I)dx^I$, which is an $\mathbb{C}$-valued p-form, then $tr([\omega, \nu]) = 0$, so

$$tr(d_A\omega) = tr(d\omega) = dtr(\omega) \tag{1.12}$$

define the *k-th Chern form* to be $tr(\underbrace{F \wedge \ldots \wedge F}_{k}) = tr(F^k)$. It is closed since

$$dtr(F^k) = tr(d_AF) = 0 \tag{1.13}$$

so it defines a cohomology class $c_k(E)$ called the *k-th Chern class*. We write $c_k(E)$ instead of $c_k(A)$ because under a perturbation of A, $tr(F^k)$ changes by an exact form:

$$\begin{aligned} \delta tr(F^k) &= ktr(\delta F \wedge F^{k-1}) = ktr(d_A(\delta A) \wedge F^{k-1}) = ktr(d_A(\delta A \wedge F^{k-1})) \\ &= d[ktr(\delta A \wedge F^{k-1})] \end{aligned} \tag{1.14}$$

since every connection for a fixed vector bundle E can be deformed into one another via a continuous path[2], they all give the same c_k. So c_k only depends on the vector bundle E. Let $L_k := tr(F^k)$, $S_k := \int_{M^4} tr(F^k)$. If M^4 has no boundary then by generalized Stoke's theorem

$$\delta S_k = \int_{\partial M^4} ktr(\delta A \wedge F^{k-1}) = 0$$

[1] Recall for example we can write down a connection for each of the two charts around a magnetic monople, and they differ by a gauge transformation.

[2] For example, $[A_1 \to A_2](t) := tA_1 + (1-t)A_2$, which is well-defined on $U_\alpha \cap U_\beta$ because $\widehat{g}_{\alpha\beta}[A_1 \to A_2] = [\widehat{g}_{\alpha\beta}A_1 \to \widehat{g}_{\alpha\beta}A_2]$

so S_k depends on E, not A. For the trivial bundle $E = M^4 \times V$ we can choose $A \equiv 0$ everywhere, so $S_k = 0$.

It turns out that[4] for any vector bundle E,

$$\frac{1}{k!}(\frac{i}{2\pi})^k S_k \in \mathbb{Z} \tag{1.15}$$

1.2.2 Chern-Simons Action

Let $k = 2$. Consider a 4-manifold with boundary, with the trivial bundle (so we can have a globally defined connection). Another way to see $S_2 = 0$ is to note that $L_2 := tr(F^2) = dL_{CS}$, where

$$L_{CS} := tr(A \wedge A + \frac{2}{3} A \wedge A \wedge A) \tag{1.16}$$

for a globally defined A. Let M^3 be a 3-manifold without boundary with the trivial bundle. Define

$$S_{CS} := \int_{M^3} L_{CS} \tag{1.17}$$

It is called the *Chern-Simons Action.* By eq(1.14), under a perturbation of the connection,

$$\delta S_{CS} = 2 \int_{M^3} tr(F \wedge \delta A) \tag{1.18}$$

To see why S_{CS} is interesting, consider a 4-manifold M^4 without boundary with the trivial bundle. We knew that $S_2 = 0$ for M^4. However, if we cut M^4 along a 3-manifold into two pieces M_1^4 and M_2^4 with M^3 as the common boundary, do a gauge transformation g on the bundle of M^3 in M_2^4, then glue the two bundles back, the resulting bundle on M_4 may not be trivial in general, and S_2 for the new bundle is the difference between $S_{CS}[\hat{g}(A)]$ and $S_{CS}[A]$. We also know from eq(1.15) that $S_2 = 8\pi^2 n$. We elucidate the above discussion with the following claim.

Claim. Let g be a gauge transformation on the trivial bundle on M^3, then

$$\delta S_{CS} := S_{CS}[\hat{g}[A]] - S_{CS}[A] = \begin{cases} 0 & for\ g\ connected\ to\ the\ identity \\ 8\pi^2 n\ for\ some\ n \in \mathbb{Z} & for\ general\ g \end{cases}$$

Proof. Suppose $g(p)$ is connected to the identity. Let $g(p,s)$ be the path from $g(p)$ to identity, ie. $g(p,0) = Id$ and $g(p,1) = g(p)$. Construct $M^3 \times [0,1]$ with the trivial bundle and two 1-forms on $M^3 \times [0,1]$:

$$A_1(p,s) := \hat{g}(p,s)A(p) \tag{1.19}$$

$$A_2(p,s) := A(p) \tag{1.20}$$

Next we identify $M^3 \times \{0\}$ with $M^3 \times \{1\}$ to obtain $M^3 \times S^1$, and identify $A_i(p,0)$ with $A_i(p,1)$ for $i = 1,2$. For A_1, we have

$$\int_{M^3 \times S^1} L_2^{(1)} = \int_{M^3 \times \{1\}} L_{CS}[A_1] - \int_{M^3 \times \{0\}} L_{CS}[A_1] = \delta S_{CS} \tag{1.21}$$

For A_2,

$$\int_{M^3 \times S^1} L_2^{(2)} = \int_{M^3 \times \{1\}} L_{CS}[A_2] - \int_{M^3 \times \{0\}} L_{CS}[A_2] = 0 \tag{1.22}$$

but since L_2 is independent of gauge, and A_1, A_2 differs by a gauge, $L_2^{(1)} = L_2^{(2)}$, so $\delta S_{CS} = 0$

For general $g(p)$ replace eq(1.19) with $A_1(p,s) := s\hat{g}(A(p)) + (1-s)A(p)$. Eq(1.21) still holds, so

$$\delta S_{CS} = \int_{M^3\times S^1} L_2^{(1)} = 8\pi^2 n \tag{1.23}$$

by eq(1.15) with $M^4 = M^3 \times S^1$. □

2 Chern-Simons Theory and Knot Invariants

Consider a quantum field theory with the Chern-Simons Action

$$S = \frac{k}{4\pi} S_{CS} = \frac{k}{4\pi}\int_{M^3} tr(A\wedge A + \frac{2}{3}A\wedge A\wedge A) \tag{2.1}$$

where $k \in \mathbb{Z}$ is the *level* of the theory. By (1.23), $\delta S = 2\pi kn$ under gauge transformation. So e^{iS} is gauge invariant.

Suppose there are some non-intersecting knots $C_1,\ldots,C_r$ in M^3. For each C_i, we assign an irreducible representation (irr rep) R_i of G, and construct a Wilson loop $W_{R_i}(C_i) = Tr_{R_i}(Pe^{\oint_{C_i} A})$. Consider the vev for the product of all W_{R_i}'s,

$$Z(M^3,(C_1,R_1),\ldots,(C_r,R_r)) =< \prod_{i=1}^r W_{R_i}(C_i) >= \int_{\mathcal{A}/\mathcal{G}} [DA]\prod_{i=1}^r W_{R_i}(C_i)e^{i\frac{k}{4\pi}S_{CS}[A]} \tag{2.2}$$

where the integral is carried over the space of all connections $\mathcal{A}$ modulo gauge transformations $\mathcal{G}$. Since S_{CS} does not depend on the metric, (2.2) is invariant under diffeomorphism of M^3 (also known as "general covariance" in physics). So (2.2) is a knot invariant for each group G and each assignment R_i to the link components. In the case where no Wilson loop is inserted, (2.2) gives an invariant[3] of the 3-manifold M^3. In the rest of this essay, we compute (2.2) and see its implications. We will see $G = U(1)$ corresponds to the linking number and $G = SU(2)$ with the fundamental representation corresponds to the Jones polynomial.

2.1 Large k limit, $G = U(1)$

Take $M^3 = S^3$, $G = U(1)$. In the large k limit, the phase in (2.2) fluctuates rapidly. The integral in dominated by stationary points of $S_{CS}[A]$, i.e. the classical solution, which is $F = 0$ from eq(1.18). Since $U(1)$ is abelian, $A\wedge A = 0$. Let $A = iA_j dx^j$, and $B_k = \epsilon_{ijk}\partial_i A_j$ where $A_j, B_j \in \mathbb{C}$, then

$$S_{CS} = \int_{S^3} A\wedge dA = -\int_{S^3} \vec{A}\cdot\vec{B}d^3x \tag{2.3}$$

We assign a $\frac{k}{4\pi}$ $U(1)$-charge to all loops, so $W(C_i) = e^{i\frac{k}{4\pi}\oint_{C_i} A_j dx^j}$. Then we will see inserting $W(C_i)$ is the same as imposing the condition that there is a unit flux of $\vec{B}$ in each wire C_i. Compute

$$[\prod_i W(C_i)]e^{\frac{ik}{4\pi}S_{CS}} = e^{\frac{ik}{4\pi}(\sum_i \oint_{C_i}\vec{A}\cdot d\vec{x} + S_{CS})}$$

so the stationary phase condition becomes

$$\frac{\delta}{\delta A}[\sum_i \oint_{C_i} \vec{A}\cdot d\vec{x} + S_{CS}] = 0 \tag{2.4}$$

[3] the vev obtained depends on the "framing" of the 3-manifold.

From (2.3),

$$\frac{\delta S_{CS}}{\delta \vec{A}(x)} = -\vec{B}(x) \tag{2.5}$$

whereas

$$\frac{\delta}{\delta \vec{A}(x)}[\oint_{C_i} \vec{A}\cdot d\vec{x}] = \vec{j}_{(i)}(x) := \int d\tau \frac{\partial \vec{\lambda}}{\partial \tau}\delta^3(x-\vec{\lambda}(\tau)) \tag{2.6}$$

where $\vec{\lambda}$ is a parametrization of C_i, $\vec{j}_{(i)}(x)$ is seen to be a unit current running in the wire C_i.

(2.4),(2.5),(2.6) yields

$$\vec{B} = \sum_{i=1}^{r} \vec{j}_{(i)}(x) \tag{2.7}$$

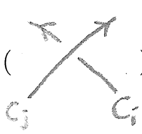
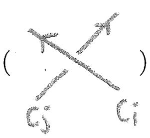
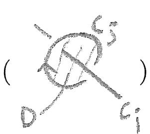
Let $\epsilon_{ijk}\vec{j}_{(k)}(x)dx^i \wedge dx^j = dA_{(i)}$ for some $A_{(i)} = \vec{A}_{(i)} \cdot d\vec{x}$. This does not fix $A_{(i)}$, but we will see the choice of $A_{(i)}$ is arbitrary.

(2.3) becomes

$$S_{CS} = -\int_{S^3} d^3x[\sum_{i\neq j} \vec{A}_{(i)}\cdot \vec{j}_{(j)} + \sum_i \vec{A}_{(i)}\cdot \vec{j}_{(i)}] \tag{2.8}$$

The choice of $A_{(i)}$ is arbitrary because suppose $dA'_{(i)} = dA_{(i)}$, then since S^3 is simply connected, every closed 1-form is exact. so $A'_{(i)} - A_{(i)} = dC$ for some 0-form C. then $\int_{S^3} d^3x\vec{A'}_{(i)}\cdot\vec{j}_{(j)} - \int_{S^3} d^3x\vec{A}_{(i)}\cdot\vec{j}_{(j)} = \oint_{C_j}(\vec{A'}_{(i)} - \vec{A}_{(i)})\cdot d\vec{x} = \oint_{C_j} dC = 0$.

Consider the term with $i \neq j$. $\int_{S^3} d^3x\vec{A}_{(i)}\cdot\vec{j}_{(j)} = \oint_{C_j}\vec{A}_{(i)}\cdot d\vec{x}$. We show it is $L(C_i, C_j)$ in eq(1.2)[4]. Let $L_{i,j} = \oint_{C_j}\vec{A}_{(i)}\cdot d\vec{x}$. Then

$L_{i,j}($  $) - L_{i,j}($  $) = L_{i,j}($  $) = \int_D dA_{(i)} = \int_D \vec{j}_{(i)}\cdot d\vec{n} = 1$

So $L_{i,j}$ transforms in the same way as $L(C_i, C_j)$ under flipping sign of crossings. Starting with a link diagram, change the sign of crossings until C_i, C_j are unlinked from each other, in which case we can throw one link infinitely away from the other, so $L_{i,j} = 0$. So $L_{i,j}$ has the same initial value as $L(C_i, C_j)$. So $L_{i,j} = L(C_i, C_j)$.

Now consider the term with $i = j$. We now show $\int_{S^3}\vec{A}_{(i)}\cdot\vec{j}_{(i)}$ is w_i in eq(1.2). The first thing to notice is that $\int_{S^3}\vec{A}_{(i)}\cdot\vec{j}_{(i)}$ is ill-defined, as both $\vec{A}_{(i)}$ and $\vec{j}_{(i)}$ are singular on C_i. We attempt to regularize it by replacing $\vec{j}_{(i)}$, a flux in a 1-demsional wire C_i of zero thinkness, with $\vec{J}_{(i)}$, a flux in a tube of some finite radius, with C_i as its core.

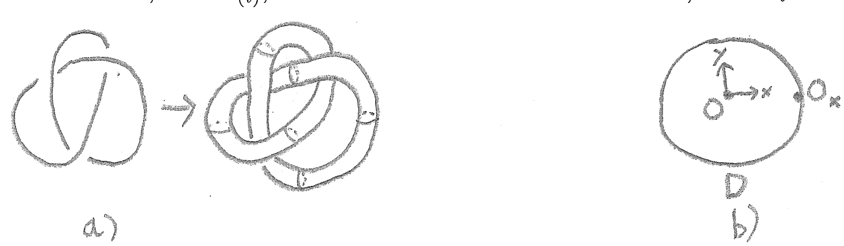
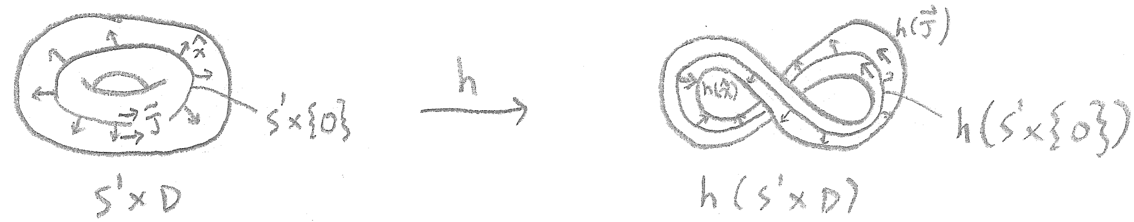


Fig. 8: a) Thickening up C_i into its tubular neighbourhood. b) Parametrizing D

Let $S^1 \times D$ be a solid torus parametrized by θ, x, y, with θ running in the S^1 direction. D is a disc with radius r and center O. Let $\vec{J} = (\pi r^2)^{-1}\hat{\theta}$ be a uniform flux with total current 1 in $S^1 \times D$. Let $h : S^1 \times D \to S^3$ be an embedding such that $h(S^1 \times \{O\}) = C_i$. Then $h(\hat{x})$ is a framing of C_i. h maps the flux $\vec{J}$ to $\vec{J}_{(i)} := h(\vec{J})$.

[4] If we treat $\vec{A}$ as the "magnetic field" generated by the current $\vec{j}$, this term the line integral along C_j of magnetic field around a wire C_i carrying unit current, which is easily seen to be the linking number between the two knots.



To calculate $\int_{S^3}\vec{A}_{(i)}\cdot\vec{J}_{(i)}$, where $\epsilon_{ijk}\vec{J}_{(k)}(x)dx^i\wedge dx^j = dA_{(i)}$, we subdivide D into N smaller discs $\{d_\alpha : \alpha = 1\dots N\}$, so $S^1\times D$ is subdivided into N small cables $\{S^1\times d_\alpha : \alpha = 1\dots N\}$.

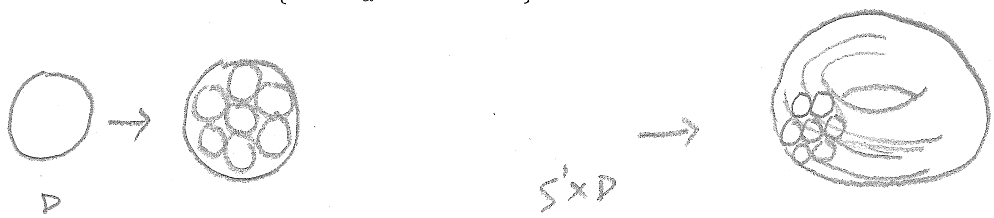


Fig. 9: Cabling of $S^1\times D$

Let $\vec{J}_\alpha$ be the flux in the tube $S^1\times d_\alpha$. Let $\vec{J}_{(i),\alpha} := h(\vec{J}_\alpha)$, and $\epsilon_{ijk}\vec{J}_{(k),\alpha}(x)dx^i\wedge dx^j = dA_{(i),\alpha}$. Let $C_{i,\alpha} = h(S^1\times\{O_\alpha\})$, where O_α is the core of d_α. Then for $\alpha\neq\alpha'$, $L(C_{i,\alpha},C_{i,\alpha'})$ does not depend on which two points O_α,$O_{\alpha'}$ are chosen in D, since moving O_α around in D is the same as moving the knot $h(S^1\times\{O_\alpha\})$ around in S^3 without crossing $h(S^1\times\{O_{\alpha'}\})$, so their linking number is unchanged. So we can pick $O_\alpha = O$,$O_{\alpha'} = O_x$ as indicated in Figure 8, so $L(C_{i,\alpha},C_{i,\alpha'}) = w_i$, where w_i is the self-linking number of C_i with framing $h(\hat{x})$. Next, write

$$\int_{S^3}\vec{A}_{(i)}\cdot\vec{J}_{(i)} = \int_{S^3}[\sum_{\alpha\neq\alpha'}\vec{A}_{(i),\alpha}\cdot\vec{J}_{(i),\alpha'} + \sum_{\alpha}\vec{A}_{(i),\alpha}\cdot\vec{J}_{(i),\alpha}]$$

and let $N\to\infty$. $\sum_{\alpha\neq\alpha'}\propto N(N-1)$,$\sum_\alpha\propto N$, $\vec{A}_{(i),\alpha}\propto 1/N$, $\vec{J}_{(i),\alpha'}\propto 1/N$, by the previous case with $i\neq j$, and the discussion in the previous paragraph for $\alpha\neq\alpha'$, $\int_{S^3}\vec{A}_{(i),\alpha}\cdot\vec{J}_{(i),\alpha'}\approx\frac{1}{N^2}L(C_{i,\alpha},C_{i,\alpha'}) = \frac{1}{N^2}w_i$. The first term goes as $N(N-1)\int_{S^3}\vec{A}_{(i),\alpha}\cdot\vec{J}_{(i),\alpha'}\approx\frac{N(N-1)}{N^2}w_i\to w_i$. The second term $\propto\frac{N}{N^2}\to 0$. So $\int_{S^3}\vec{A}_{(i)}\cdot\vec{J}_{(i)} = w_i$. Notice the answer does not depend on r.

To conclude, $S_{CS} = -[\sum_{i\neq j}L(C_i,C_j)+\sum_i w_i] = -w(C_1,\dots,C_n)$, so the vev (2.2) yields $exp(-\frac{ik}{4\pi}w(C_1,\dots,C_n))$.

2.2 Quantization

Our general strategy of computing (2.2) is to develop a mechanism to chop the 3-manifold M^3 along a Riemann surface Σ, solving the individual pieces, and gluing them back together. To do this, we first have to understand the theory near the cut Σ, where M^3 looks like $\Sigma\times\mathbb{R}$.

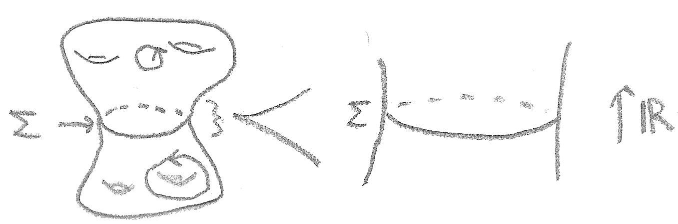


Fig. 10: Cutting M^3 along Σ

We perform canonical quantization on $\Sigma\times\mathbb{R}$, where $\mathbb{R}$ is treated as the time ($x^0 = t$) direction. Choosing the gauge

$A_0 = 0$, the action(2.1) becomes

$$S = -\frac{k}{4\pi}\int dt \int_\Sigma tr(\epsilon^{ij} A_i \frac{d}{dt} A_j) = \frac{k}{8\pi}\int dt \int_\Sigma \epsilon^{ij} A_i^a \frac{d}{dt} A_j^a \tag{2.9}$$

where $A_i = A_i^a T^a$, with normalization $Tr(T^a T^b) = -\frac{1}{2}\delta^{ab}$. $i, j = 1, 2$ are the space-like indices.

The Poisson brackets are

$$\{A_i^a(x), A_j^a(y)\} = \frac{4\pi}{k}\epsilon_{ij}\delta^{ab}\delta^2(x-y) \tag{2.10}$$

When no Wilson loops are intersecting Σ, from (1.18), the constraint $\frac{\delta S}{\delta A_0} = 0$ implies

$$F_{ij}^a = 0 \tag{2.11}$$

In principle, we first apply the constraint (2.11) and fix the gauge, which changes the Poisson bracket structure (2.10). Then we quantize (i.e. promote A_i^a to operators and Poisson brackets to commutators.). But it is hard to see how the Poisson bracket change due to the constraints.

Instead, we study the Hilbert space. Let M^3 be separated into two pieces M_1, M_2, with common boundary Σ. The vev is

$$\langle \psi_2 | \psi_1 \rangle = \int_{\mathcal{A}/\mathcal{G}} [DA] e^{iS[A]} = \int_{\mathcal{A}_\Sigma/\mathcal{G}} [DA_\Sigma] e^{iS'_\Sigma[A_\Sigma]} \bar{\psi}_2[A_\Sigma]\psi_1[A_\Sigma] \tag{2.12}$$

where A_Σ are connections on Σ satisfying (2.11), and $\psi_i[A_\Sigma] := \int_{\substack{\mathcal{A}_{M_i}/\mathcal{G} \\ A_{M_i}|_\Sigma = A_\Sigma}} [DA_{M_i}] e^{iS''_\Sigma[A_{M_i}]}$. In the above equation, we essentially factor the integral $\int_{\mathcal{A}/\mathcal{G}}[DA]$ over connections on the whole M^3 into an integral of connections on Σ and an integral of connections over M_1, M_2 satisfying the boundary conditions on Σ. S'_Σ and S''_Σ can be written out more explicitly in terms of S_{CS} and a metric on Σ, but it is messy and we do not do it here. The important point is that the vev can be written as an inner product in some Hilbert space $\mathcal{H}_\Sigma$, in which a vector $|\psi\rangle$ is associated with each manifold M with boundary Σ. For example if M_2 is replaced by another manifold M_2' with the same boundary, then the vev will become $\langle \psi_2' | \psi_1 \rangle$ for some $\langle \psi_2' |$ corresponding to M_2'.

2.2.1 Dimension of $\mathcal{H}_{S^2}$

As an example, we consider $\Sigma = S^2$. Since S^3 is simply connected, a holonomy on S^2 for a flat connection A does not depend on the path taken and only depends on the starting and ending point (since every path with the same starting and ending points can be deformed continuously into one another in S^3, and a holonomy with flat connection does not change under continuous deformation of path) Therefore we can do a gauge transformation to reduce every holonomy to the identity map. Since the holonomy determines the connection uniquely, the connection after the gauge transformation is simply $A \equiv 0$. Under a gauge transformation $A \to gAg^{-1} - dgg^{-1}$, the original A can therefore be parametrized by a map $g : \Sigma \to G$,

$$A_i = -(\partial_i g)g^{-1} = g\partial_i g^{-1} \tag{2.13}$$

Moreover, this means $\mathcal{A}_{S^2}/\mathcal{G}$ subject to (2.11) is a single point. So $\mathcal{H}_{S^2}$ is 1-dimensional.

2.2.2 Inclusion of Wilson lines

We would like to study how Wilson lines with irr reps R_k at points p_k on Σ affects the constraint (2.11). We expect the RHS to change into a linear combination of $\delta^2(x - p_k)$'s, as we have seen in (2.7). Since we are working in $\Sigma \times \mathbb{R}$, the

Wilson lines can be treated as straight lines running in the t direction. for $r = 1$, let $A_0 = A_0^a T^a$. Consider (2.2)

$$\begin{aligned} < W_R(C) > &= \int_{\mathcal{A}/\mathcal{G}} [DA] W_R(C) e^{iS[A]} \\ &= \int_{\mathcal{A}/\mathcal{G}} [DA] Tr_R(P e^{\int A_0 dt}) e^{iS[A]} \\ &= \int_{\mathcal{A}/\mathcal{G}} [DA] Tr_R(P e^{\int A_0^a T^a dt}) e^{iS[A]} \\ &= \int_{\mathcal{A}/\mathcal{G}} [DA] Tr_R(P e^{\int A_0^a T^a \delta^2(x-p) dx^2 dt}) e^{iS[A]} \end{aligned}$$

from (1.18), $\frac{\delta S_{CS}}{\delta A_0^a(x)} = -F_{12}^a(x)$, so $\frac{\delta S}{\delta A_0^a(x)} = -\frac{k}{4\pi} F_{12}^a(x)$. $\frac{\delta}{\delta A_0^a(x)} \int A_0^a T^a \delta^2(x-p) dx^2 dt = T^a \delta^2(x-p)$, so we may expect for $r = 1$

$$\frac{k}{4\pi} F_{12}^a(x) = (-i) \delta^2(x-p) T_R^a \tag{2.14}$$

and for general r,

$$\frac{k}{4\pi} F_{12}^a(x) = (-i) \sum_{k=1}^{r} \delta^2(x - p_k) T_{R^k}^a \tag{2.15}$$

However, this equation does not make sense because the LHS is a c-number while the RHS is an operator which does not commute with other operators. There are ways to work around this, for example by making the LHS into a quantum number. But we do not do it here.

We use (2.15) to argue that a Wilson loop intersecting Σ in the reversed direction gives the same constraint as a Wilson loop carrying a conjugate representation. Since F_{12}^a is real, taking complex conjugation of (2.15) yields

$$\begin{aligned} \frac{k}{4\pi} F_{12}^a(x) &= -(-i) \sum_{k=1}^{r} \delta^2(x - p_k) T_{R^k}^{a*} \\ \frac{k}{4\pi} F_{21}^a(x) &= (-i) \sum_{k=1}^{r} \delta^2(x - p_k) T_{R^k}^{a*} \end{aligned} \tag{2.16}$$

Reversing the direction of the Wilson loop is the same is reversing the orientation on Σ, which is the same as interchanging x^1, x^2. $T_{R^k}^{a*}$ are the generator for the conjugate representation. We can therefore treat all Wilson lines as going in the same direction through Σ, taking conjugation of irr reps if necessary.

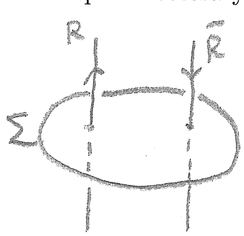


Fig. 11:

2.3 Hilbert Spaces

To prepare for our calculations in the next few sections, we study the Hilbert spaces for the sphere and the torus, $\mathcal{H}_{S^2}$ (with Wilson loop crossing) and $\mathcal{H}_{T^2}$ (without Wilson loop crossing).

2.3.1 $\Sigma = S^2$, with Wilson loops crossing

For S^2 with Wilson loops C_k, with representations R_k, $k = 1, \ldots, r$ going in the positive t direction through Σ, Witten[1] argued that

Fact 1. *In the $k \to \infty$ limit,*

$$\mathcal{H}_{S^2} = Inv(\otimes_{i=1}^{r} R_i) \tag{2.17}$$

which is the G-invariant subspace in the tensor product $\otimes_{i=1}^{r} R_i$. In the case k is finite, $\mathcal{H}_{S^2}$ is a subspace of (2.17).

It turns out that there is a close relationship between CS theory and Wess-Zumino-Witten (WZW) model, a 2D CFT on Σ. $\mathcal{H}_\Sigma$ is related to conformal blocks in WZW model and $dim(\mathcal{H}_{S^2})$ =number of conformal blocks in WZW.

As an application of Fact 1, we calculate $dim(\mathcal{H}_{S^2})$ for the cases that we will need to use later:

(i) No Wilson line: $dim\,(H_{S^2}) = 1$, as derived in 2.2.1

(ii) One Wilson line (R): $dim\,(H_{S^2}) = \begin{cases} 1 & if\ R = the\ trivial\ representation \\ 0 & else \end{cases}$

(iii) Two Wilson lines (R_i, R_j): $dim\,(H_{S^2}) = \begin{cases} 1 & for\ R_i = \overline{R_j} \\ 0 & else \end{cases}$

(iv) Four Wilson lines, with two incoming and two outgoing lines carrying the same irr rep (R,R,$\bar{R}$,$\bar{R}$): $dim\,(H_{S^2}) = s$, where $R \otimes R = \sum_{i=1}^{s} E_i$ for irr reps E_i.

In particular, for the fundamental representation of $SU(N)$, $s = 2$ ($\because \square \otimes \square = \square\square \oplus \begin{smallmatrix}\square \\ \square\end{smallmatrix}$)

2.3.2 $\Sigma = T^2$, no Wilson loops crossing

As stated in [1], Verlinde[3] showed that, if we chose the two cycles a and b as the basis for $H^1(\Sigma, \mathbb{Z})$, there is a corresponding choice of basis in $\mathcal{H}_{T^2}$, constructed as follows. First let T^2 bounds a solid torus $\bar{T} = S^1 \times D^2$. Let a Wilson loop carrying representation $R_i \in A_k$ (to be defined below) run inside the solid torus, going in the non-contractible direction once (and equipped with a framing for which the self-linking number is 0):

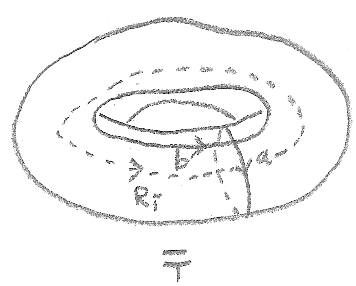


Fig. 12: $|v_i\rangle$

The path integral inside this solid torus gives the vector $|v_i\rangle$ in $\mathcal{H}_{T^2}$.

A_k is defined as follows. Consider the level k, integrable highest weight representations of the loop group LG of G. There are finitely many such reps if k is finite. In each such representation, the highest weight space gives an irr rep of G. A_k is the collection of these irr reps. In particular it means $\mathcal{H}_{T^2}$ is finite dimensional. For instance, when $G = SU(2)$, A_k is the collection of spin s representations, where $s \leq \frac{k}{2}$. The trivial representation, R_0 corresponds to the case when there are no Wilson loops in $\bar{T}$. In general A_k always contains the trivial representation. For $SU(N)$, the fundamental representation is in A_k for $k \geq 1$.

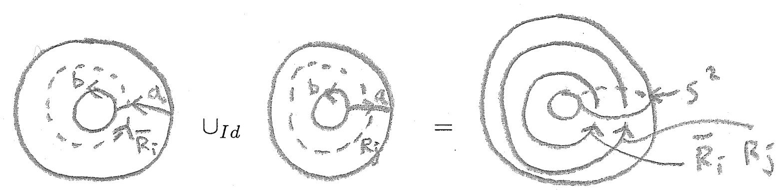
Fact 2. *$\mathcal{H}_{T^2} = span\{|v_i\rangle\}$, where $\{|v_i\rangle\}$ are constructed as above.*

We will use Fact 1(in the next subsubsection) and Fact 2 (in 2.4.2) without proving them.

2.3.3 Applications of Fact 1

2.3.3.1 $\langle v_i|v_j\rangle = \delta_{ij}$

To calculate $\langle v_i|v_j\rangle$, we glue together two solid tori with the identity map between the two torus on their surfaces.



$\bar{T}$ $\bar{T}$ $S^2 \times S^1$

Since a conjugation is performed on $\langle v_i|$, the orientation on its surface is reversed and the representation on the Wilson loop is conjugated. At a cross-section of the solid tori, $D^2 \cup_{Id} D^2 = S^2$, so $\bar{T} \cup_{Id} \bar{T} = S^2 \times S^1$.

$$\begin{aligned}\langle v_i|v_j\rangle &= Z(S^2 \times S^1, (C_i, R_i), (C_j, \bar{R}_j)) = tr_{\mathcal{H}_{S^2}}(e^{-iHt}) = tr_{\mathcal{H}_{S^2}}(1) \\ &= dim(\mathcal{H}_{S^2}) = \begin{cases} 1 & for\ R_i = R_j \\ 0 & else \end{cases} = \delta_{ij} \end{aligned} \tag{2.18}$$

Recall how Z was defined in (2.2). In the second step we treated S^1 as $[0, t]$ with field configurations identified at $S^1 \times \{0\}$ and $S^1 \times \{t\}$. H is the hamiltionian of the system. In the third step we used $H = 0$ which can be calculated from (2.9). This reflects the topological nature of the theory since the vev should not depend on t, the length of S^1. The fifth step used Fact 1.

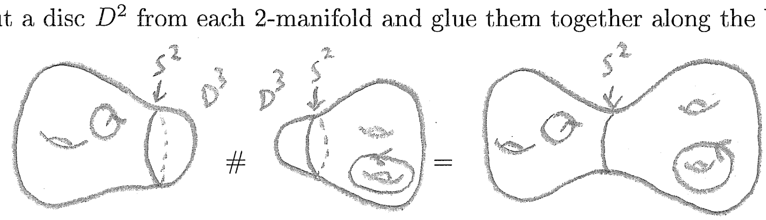
2.3.3.2 $Z(S^2 \times S^1)$

Putting $R_i = R_j = R_0$ (the trivial representation) in (2.18) yields

$$Z(S^2 \times S^1) = 1 \tag{2.19}$$

2.3.3.3 Connected sum of manifolds

Given 3-manifolds M_1, M_2, a connected sum $M_1 \# M_2$ is obtained by first removing a copy of D^3(a solid ball) from each of them to obtain two manifolds $\bar{M}_1$, $\bar{M}_2$ with boundary $\partial\bar{M}_1 = \partial\bar{M}_2 = \partial D^3 = S^2$. Then glue the two manifolds together by identifying the two S^2's with reversed orientation from one of them, the result is $M_1 \# M_2$. It is useful to picture the 2D analog, where we cut out a disc D^2 from each 2-manifold and glue them together along the boundary S^1.



M_1 M_2 $M_1 \# M_2$

by (i) of Fact 1, if there are no Wilson loops passing through S^2, $dim\,(H_{S^2}) = 1$. So for any $|a\rangle\,, |b\rangle\,, |c\rangle\,, |d\rangle \in H_{S^2}$, $\langle a|b\rangle \langle c|d\rangle = \langle a|d\rangle \langle c|b\rangle$,

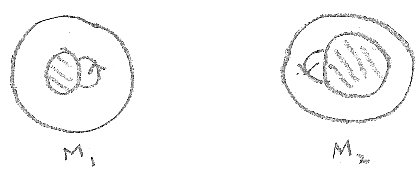
$$Z(M_1 \# M_2) Z(S^3) = \langle \bar{M_1}|\bar{M_2}\rangle \langle D^3|D^3\rangle = \langle \bar{M_1}|D^3\rangle \langle D^3|\bar{M_2}\rangle = Z(M_1)Z(M_2) \tag{2.20}$$

where we used $\bar{S^3} = D^3$ and $S^3 \# S^3 = S^3$. Denote $\langle M\rangle := \frac{Z(M)}{Z(S^3)}$, then

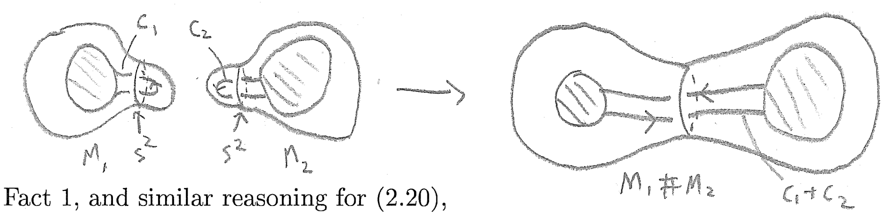
$$\langle M_1 \# M_2\rangle = \langle M_1\rangle \langle M_2\rangle \tag{2.21}$$

2.3.3.4 Connected sum of knots

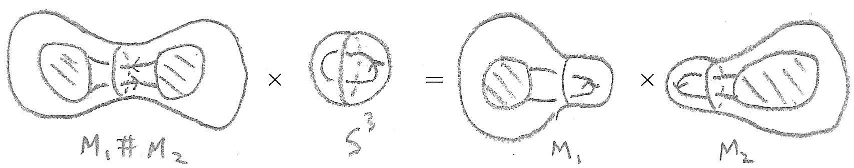
Suppose we have two Wilson loops C_1 and C_2 inside M_1 and M_2 respectively, carrying the same irr rep R:



We choose S^2 to cutout a segment of Wilson line and form the connected sum $(M_1 \# M_2, C_1 + C_2)$



by (iii) of Fact 1, and similar reasoning for (2.20),



$$Z(M_1 \# M_2, C_1 + C_2) Z(S^3, O) = Z(M_1, C_1) Z(M_2, C_2) \tag{2.22}$$

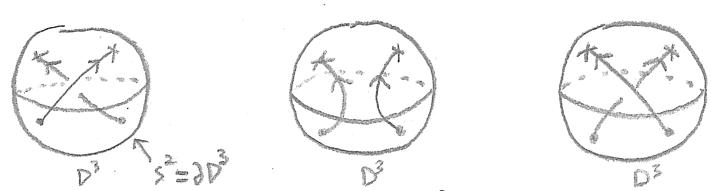
where O represents the unknot. Denote $\langle M, C\rangle := \frac{Z(M,C)}{Z(S^3,O)}$, then

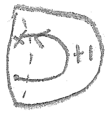
$$\langle M_1 \# M_2, C_1 + C_2\rangle = \langle M_1, C_1\rangle \langle M_2, C_2\rangle \tag{2.23}$$

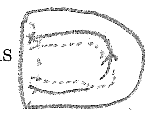
if $M_1 = M_2 = S^3$, then $\langle S^3, C_1 + C_2\rangle = \langle S^3, C_1\rangle \langle S^3, C_2\rangle$. Compare to (1.10).

2.3.3.5 Skein relation, Part 1

Recall a skein relation relates knot invariants between knots that differ only at a crossing, where they take values L_+, L_0 and L_-. Consider the following three diagrams of manifolds with boundary:

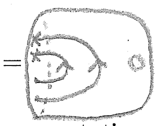


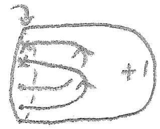
They give rise to three vectors in $\mathcal{H}_{S^2}$, the Hilbert space of S^2 with 4 Wilson lines crossing through. It turns out the vectors depend on the self-linking number. So we choose the framing in each diagram so that their contribution to the self-linking number is 0. We denote the self-intersection number with respect to that of the blackboard framing with a number. For example 

means , where the dotted line is the framing of the solid line.

Let $|L_+\rangle$, $|L_0\rangle$, $|L_-\rangle$ denote the three states

$|L_+\rangle =$ 

, $|L_0\rangle =$ 

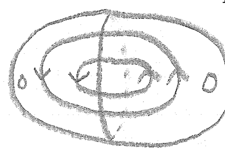
, $|L_-\rangle =$ 

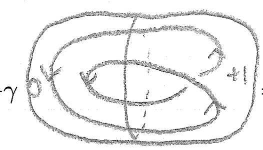
and suppose the Wilson loops carry the fundamental representation R of $SU(N)$. By (iv) of Fact 1, the Hilbert space where these states live is at most two-dimensional. So these three states must be linearly dependent. $\exists \alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\alpha\,|L_+\rangle + \beta\,|L_0\rangle + \gamma\,|L_-\rangle = 0 \tag{2.24}$$

We see that α, β, γ are the coefficients in the skein relation. Since α, β, γ are all up to a constant factor, the ratios between them are two unknowns. We multiply (2.24) with two bra vectors to produce two equations:

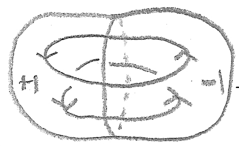
α 

$+\beta$ 

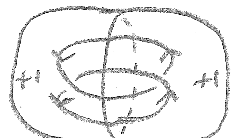
$+\gamma$ 

$= 0$ (2.25)

I, II, III

α 

$+\beta$ 

$+\gamma$ 

$= 0$ (2.26)

IV, V, VI

Now to derive the skein relation we compute these 6 diagrams and solve for the ratio between α, β, γ. We note that $I = III = V = Z(S^3, O)$. II and IV are the connected sum of two S^3 with an unknot. By (2.20), $II = IV = \frac{[Z(S^3,O)]^2}{Z(S^3)}$.

2.4 Framing dependence and surgery

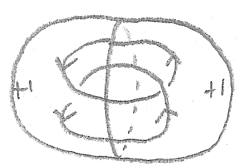
In this subsection we answer the remaining questions: How to compute $Z(S^3)$, $Z(S^3, O)$ and VI? How does the vev depend on the framing? We turn to the last question.

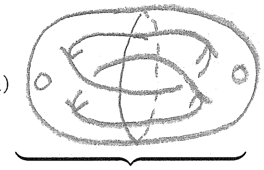
2.4.1 Framing dependence

Fact 3. *If the self-intersecting number is increased by 1, then its vev is multiplied by* $e^{2\pi i h_R}$*, where* R *is the representation of the Wilson loop concerned, and* $h_R = \frac{\lambda\cdot(\lambda+2\rho)}{2(k+g)}$ *is the conformal weight of a primary field in WZW model. Here* λ *is the highest weight in the irr rep* R*,* ρ *is the Weyl vector for* G *and* g *is the coxeter number of* G*. (The normalization is* $|\theta|^2 = 2$*, where* θ *is the longest root)*

For $SU(N)$, $g = N$, for its fundamental representation, $\lambda \cdot (\lambda + 2\rho) = \frac{N^2-1}{N}$. So $h_R = \frac{N^2-1}{2N(N+k)}$. The derivation for the conformal weight of a primary field in WZW model can be found in [5]. Although we will not prove this fact, we can try to understand it by thinking of Wilson lines as the trajectory of a particle with fractional statistics in a 2+1 dimensional theory, where h_R is the "spin" of the particle. Cutting the trajectory, rotating one part by 2π and gluing them back increases the self-linking number by 1, and its effect on the vev is same as rotating the particle in the 2D universe by 2π.

$e^{2\pi i h_R}$ is the phase factor acquired by this transformation. So

$VI =$ 

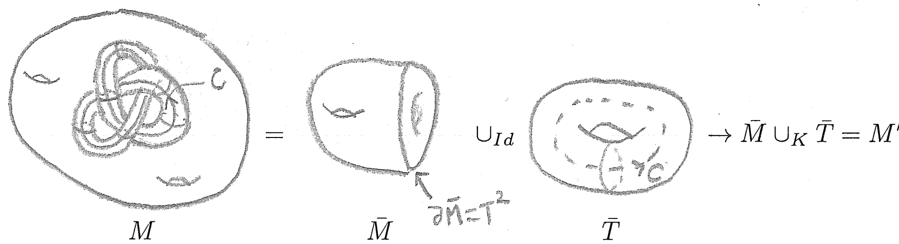
$= e^{+2(2\pi i h_R)}$ 

$\underbrace{}_{VI'}$ (2.27)

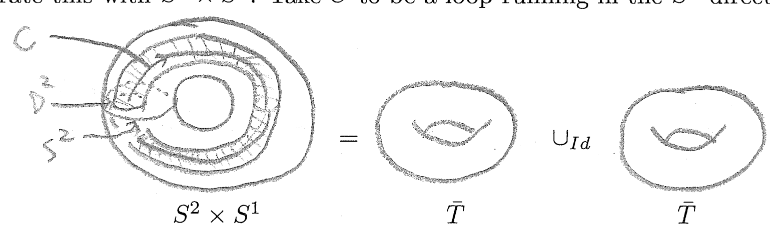
2.4.2 Surgery on 3-manifolds

Let C be a loop (not necessarily a Wilson loop) situated in a 3-manifold M. Cutting M along the boundary of a tubular neighbourhood of C yields $\bar{M}$ (M without the tube) and $\bar{T}$ (a solid torus). Both $\partial\bar{M}$ and $\partial\bar{T}$ are diffeomorphic to T^2. Denote them $T^2_{(1)}$ and $T^2_{(2)}$ respectively.

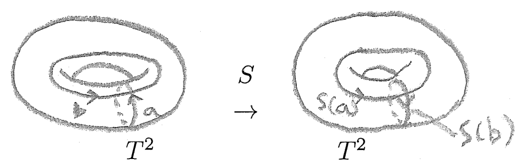
If we construct an automorphism of T^2, $K : T^2 \to T^2$, and glue together $T^2_{(1)}$ and $T^2_{(2)}$ by identifying p with $K(p)$ for every $p \in T^2_{(1)}$, we obtain another manifold M' without boundary. The above procedure is known as a *surgery* on M.



If $K = Id$, then we just glue them back the same way it was cut open, so $M = M'$. It is known that every closed, orientable, connected 3-manifold can be reduced to S^3 via a sequence of surgeries (also known as the Lickorish–Wallace theorem). Let's illustrate this with $S^2 \times S^1$. Take C to be a loop running in the S^1 direction

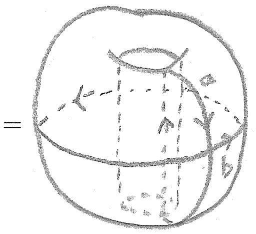


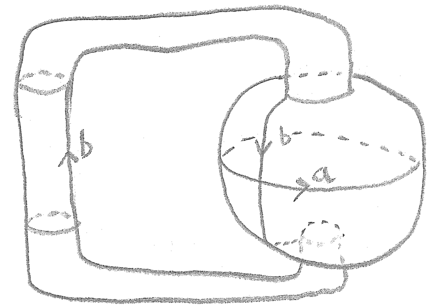
Since $S^2 - D^2 = D^2$, if we cut out the tubular neighbourhood of C, then the remaining piece in $S^2 \times S^1$ is $D^2 \times S^1 = \bar{T}$. Now we perform a "S-transformation" on T^2. Let a and b be cycles of T^2 as shown



An *S-transformation* maps $a \to b$, $b \to -a$. So if a is a contractible cycle in $T^2 = \partial\bar{T}$, then $S(a)$ will be a non-contractible cycle.

If we let $K = S$, then the contractible cycle in $T^2_{(1)}$ will be glued to the non-contractible cycle in $T^2_{(2)}$. The result is S^3.

$\bar{T} \cup_K \bar{T} =$ 

$\cup$ 

$\bar{T}$ $\qquad\qquad$ $\bar{T}$

$=$  $\cup_{Id}$  $= S^3$

D^3 $\qquad\qquad$ D^3

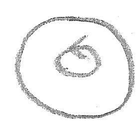
The S-transformation is a diffeomorphism on T^2. It induces an isomorphism on $\mathcal{H}_{T^2}$. By Fact 2 we express this isomorphism S by its matrix elements S_{ij} in the basis $\{|v_i\rangle\}$.

$$S\,|v_i\rangle = \sum_j S_{ij}\,|v_j\rangle$$

Since we obtained S^3 by gluing two empty $\bar{T}$'s with the S-transformation,

$$Z(S^3) = \langle v_0|S|v_0\rangle = S_{00} \tag{2.28}$$

If we glue together an empty $\bar{T}$ with a $\bar{T}$ that has a Wilson loop carrying the fundamental rep R_1 of $SU(N)$ with the S-transformation instead, we get (S^3, O), the three-sphere with an unknot.

$Z(S^3, O) =$  $= \langle v_0|S|v_1\rangle = S_{01}$ $\qquad$ (2.29)

S^3

If both $\bar{T}$'s contain a Wilson loop carrying R_1, then the result of gluing is S^3 containing a Hopf link, with linking number -1, which is exactly VI' in (2.27)

$VI' =$  $= \langle v_1|S|v_1\rangle = S_{11}$ $\qquad$ (2.30)

S^3

To find the values of S_{ij}, we quote a fact from Witten[1]. It turns out that S_{ij} is the matrix by which S is represented on the characters of the irreducible level k representations of the loop group LG. Here is a formula from the study of affine Lie algebra[5].

Fact 4. $S_{\hat{\lambda}\hat{\mu}} = C_{G,k}\sum_{\omega\in W}\epsilon(\omega)exp(\frac{-2\pi i[\omega(\lambda+\rho)\cdot(\mu+\rho)]}{k+g})$, *where* $\hat{\lambda},\hat{\mu}$ *are the representations with highest weight* λ,μ *respectively.* C_G *is a constant depending on* k*, the group* G*, and independent of* $\hat{\lambda},\hat{\mu}$*.* W is the Weyl group of G.

For $SU(2)$,

$$S_{ij} = \sqrt{\frac{2}{k+2}}\sin(\frac{\pi(i+1)(j+1)}{k+2}) \tag{2.31}$$

Reduction of vev for general 3-manifolds

Suppose M' is obtained from M be doing a surgery along C using diffeomorphism K. Then the vev for M' can be expressed in terms of K_{ij} and the vev for M with Wilson loops inserted in the tube where the surgery is performed.

$$Z(M') = \langle\psi|K|v_0\rangle = \sum_i K_{0i}\langle\psi|v_i\rangle = \sum_i K_{0i}Z(M, C_i, R_i)$$

since every closed, orientable, connected 3-manifold can be reduced to S^3 via surgeries, the vev (2.2) for any manifold with any Wilson loop can be reduced to the vev in S^3 with some Wilson loops and matrix elements of K.

The automorphism group of T^2 has many connected components. In the component containing Id the matrix for K is identity, since these diffeomorphisms can be induced from diffeomorphisms of the surrounding 3-manifold. The automorphism group modulo the group connected to the identity is called the *mapping class group* of T^2, and is generated by two elements S and T (which maps $a \to a$, $b \to b+a$, ie. cut the torus along a, twist it, and glue it back). So to study the matrix elements of a general K we only need to study S_{ij} and T_{ij}. T is related to changing the framing of the 3-manifold. We will not continue this discussion any further in this essay and refer interested reader to [1].

2.5 Skein relation, Part 2

Consider $G = SU(2)$, let Wilson loops carry the fundamental (spin 1/2) representation. We now assemble the pieces.

From (2.25), (2.26), (2.27), Fact 3, (2.28), (2.29), (2.30),

$$\alpha\frac{S_{01}}{S_{00}} + \beta(\frac{S_{01}}{S_{00}})^2 + \gamma\frac{S_{01}}{S_{00}} = 0 \tag{2.32}$$

$$\alpha(\frac{S_{01}}{S_{00}})^2 + \beta(\frac{S_{01}}{S_{00}}) + \gamma exp(2\frac{2\pi i(3)}{4(k+2)})\frac{S_{11}}{S_{00}} = 0 \tag{2.33}$$

from (2.31), and let $q = exp(\frac{2\pi i}{k+2})$, $\frac{S_{01}}{S_{00}} = \frac{\sin(\frac{2\pi}{k+2})}{\sin(\frac{\pi}{k+2})} = \frac{q-q^{-1}}{q^{1/2}-q^{-1/2}}, \frac{S_{11}}{S_{00}} = \frac{\sin(\frac{4\pi}{k+2})}{\sin(\frac{\pi}{k+2})} = \frac{q^2-q^{-2}}{q^{1/2}-q^{-1/2}}, exp(2\frac{2\pi i(3)}{4(k+2)}) = q^{3/2}$.

Setting $\beta = q^{1/2} - q^{-1/2}$ and solving, the result is $\alpha = -q$, $\gamma = q^{-1}$. So (2.24) reads

$$-q|L_+\rangle + (q^{1/2} - q^{-1/2})|L_0\rangle + q^{-1}|L_-\rangle = 0 \tag{2.34}$$

replacing variable $q^{1/2} = -t^{-1/2}$,

$$-t^{-1}|L_+\rangle + (t^{1/2} - t^{-1/2})|L_0\rangle + t|L_-\rangle = 0 \tag{2.35}$$

Recall $\langle S^3, C\rangle = \frac{Z(S^3,C)}{Z(S^3,O)}$, so $\langle S^3, O\rangle = 1$, and $-t^{-1}\langle S^3, L_+\rangle + (t^{1/2} - t^{-1/2})\langle S^3, L_0\rangle + t\langle S^3, L_-\rangle = 0$. Comparing with (1.7), (1.8) yields $V(C) = \langle S^3, C\rangle$, since these two conditions, along with diffeomorphism invariance, uniquely determine $V(C)$.

As a check, using (2.20) we compute

$$\left\langle S^3, C_1 \coprod C_2 \right\rangle = \frac{Z(S^3, C_1 \coprod C_2)}{Z(S^3, O)} = \frac{Z(S^3, C_1)}{Z(S^3, O)} \frac{Z(S^3, C_2)}{Z(S^3, O)} \frac{Z(S^3, O)}{Z(S^3)}$$
$$= \langle S^3, C_1 \rangle \langle S^3, C_2 \rangle (\frac{q - q^{-1}}{q^{1/2} - q^{-1/2}}) = -(t^{1/2} + t^{-1/2}) \langle S^3, C_1 \rangle \langle S^3, C_2 \rangle$$

which is (1.9).

2.6 The HOMFLY polynomial and the Kauffman polynomial

It turns out[1] that, for $SU(N)$ fundamental rep, the skein relation is

$$-q^{N/2} |L_+\rangle + (q^{1/2} - q^{-1/2}) |L_0\rangle + q^{-N/2} |L_-\rangle = 0$$

where $q = exp(\frac{2\pi i}{k+N})$. This is the skein relation of a HOMFLY polynomial.

For $SO(N)$ with the fundamental N-dimensional rep, one obtains the Kauffman polynomial, another kind of knot invariant.

3 Chern-Simons theory and WZW model

In the above discussions, we quoted four Facts from [1] related to the study of WZW model and affine Lie algebra. To make the story complete, we should provide justification for these four Facts. In this subsection we follow [7] and the discussion at the end of [1] to present two links between CS theory and WZW model, as a starting place to understand them.

We consider CS theory on $D^2 \times \mathbb{R}$. On a manifold with boundary, the variation of S reads

$$\delta S = \frac{k}{4\pi} \int_{D^2 \times \mathbb{R}} tr(\delta A \wedge dA + A \wedge d\delta A + 2\delta A \wedge A \wedge A)$$
$$= \frac{k}{4\pi} [\int_{S^1 \times \mathbb{R}} tr(\delta A \wedge A) + 2 \int_{D^2 \times \mathbb{R}} tr(\delta A \wedge F)]$$

In order for all elements in the gauge orbit to have the same equation of motion $\frac{\delta S}{\delta A_0^a} = -\frac{k}{4\pi} F_{12}^a = 0$, and also in order that S changes by a multiple of 2π under gauge transformation, we restrict our gauge transformations to be identity on the boundary $S^1 \times \mathbb{R}$. We also pick the gauge $A_o = 0$. Since D^2 is simply connected by the same arguments leading to (2.13) $A_i = -\partial_i g g^{-1}$ for some $g : D^2 \to G$. But in the interior of D^2 we can choose the gauge to set $A_i \equiv 0$, and on the boundary S^1 the parametrization $A_i = -\partial_i g g^{-1}$ has some redundancy since we can replace g by $g_0 g$ for a constant g_0 without changing A_i. Therefore if we fix a point b on S^1 we can always choose $g(b) = Id$. Therefore the phase space (flat connections modulo gauge transformations) is the set of maps from S^1 to G with b mapped to Id. This is the space of based loops LG/G. There is a natural action of the loop group LG on the phase space, so the Hilbert space should be a representation of LG, described by the affine Lie algebra.

We now also show that the action (2.1) in $D^2 \times \mathbb{R}$ is exactly the action for WZW model on the cylinder $S^1 \times \mathbb{R}$. From

(2.9),

$$
\begin{aligned}
S &= -\frac{k}{4\pi}\int dt \int_{D^2} tr(\epsilon^{ij} A_i \frac{d}{dt} A_j) \\
&= -\frac{k}{4\pi}\int dt \int_{D^2} tr(\epsilon^{ij} \partial_i g g^{-1} \frac{d}{dt}(\partial_j g g^{-1})) \\
&= -\frac{k}{4\pi}\int dt \int_{D^2} tr[\epsilon^{ij} \partial_i g g^{-1}(\partial_0 \partial_j g g^{-1} - \partial_j g g^{-1} \partial_0 g g^{-1})] \\
&= \frac{k}{4\pi}\int dt \int_{D^2} tr[\epsilon^{ij} \partial_i (g^{-1} \partial_j g g^{-1} \partial_0 g) + \epsilon^{ij} g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_0 g] \\
&= \frac{k}{4\pi}\int dt [\int_{S^1} dx^j tr(\epsilon^{ij} g^{-1} \partial_j g g^{-1} \partial_0 g) + \int_{D^2} (\epsilon^{ij} g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_0 g)] \\
&= k S_{WZW}(g)
\end{aligned}
$$

The second term in the fifth step depends only on the value of g at the boundary, up to an integer multiple of 2π, because as discussed above, the interior values of g can be gauged away, and S changes by of $2\pi k$ under gauge transformation.

Although we have not justified the four Facts, we can still see that the 3D CS theory is closely tied to the 2D WZW model. An understanding of either one provides insight into the other one.

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