Chapter synopses from *Groups, Graphs and Trees* by John Meier

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Chapter 1: Cayley's Theorems

A group action on a set X (or a topological space, vector space, group, etc.) is a homomorphism from G to the symmetry group of X, where $e \in G$ is mapped to the identity permutation, and $(gh)\cdot x = g\cdot(h\cdot x)$ for all $g, h \in G$, $x \in X$. The defining homomorphism for a given group action is called a representation of G. This representation is faithful if the homomorphism is injective. Cayley's basic theorem states that every group G can be faithfully represented as a group of permutations of the underlying set of G. The stabilizer of a point $x \in X$ is a subgroup of G. Free actions are those where the stabilizer of every $x \in X$ is trivial.

A graph is a set of vertices along with a set of edges, each of which is associated to an unordered pair of vertices. A graph is locally finite if every vertex has finitely many edges. A tree is a connected graph which has no cycles. A graph is a tree if and only if there is a unique edge path between any two vertices. A directed graph is one whose edges have an initial vertex and a terminal vertex, typically indicated by an arrow.

Cayley's Better Theorem states that every finitely generated group can be faithfully represented as a symmetry group of a connected, directed, locally finite graph—namely, the Cayley graph associated with a set of generators. For a group G with finite generating set S, the Cayley graph is constructed as follows: vertices correspond to group elements, and an edge labelled s is drawn between v_g and v_h if $h = gs$. The Cayley graph is also decorated with arrows and colours on its edges according to the generating set S. The decoration-preserving symmetry group of the Cayley graph of G is isomorphic to G (Theorem 1.51). An action on a graph $G \curvearrowright \Gamma$ is vertex transitive if for any vertices $v, w \in \Gamma$, there is an element $g \in G$ so that $g(v) = w$. The action of a group G on its Cayley graphs is vertex transitive.

A fundamental domain of a group action on a graph $G \curvearrowright \Gamma$ is a minimal closed subgraph whose orbit covers Γ. Every group action on a connected graph has a fundamental domain $\mathcal{F}$. The set of nontrivial elements g of G with the property $g \cdot \mathcal{F} \cap \mathcal{F} \neq \emptyset$ forms a generating set for G (Theorem 1.55).

Chapter 2: Groups generated by reflections

A group G is Hopfian if every surjection $G \to G$ is an isomorphism. An equivalent definition is that G is not isomorphic to any proper quotient of G. A group G is co-Hopfian if every injection $G \to G$ is an isomorphism, or equivalently, if it is not isomorphic to any proper subgroup.

The infinite dihedral group D_∞ is a group of symmetries of the Euclidean plane which can be thought of as a limit of the finite dihedral groups D_n. It is generated by 2 reflections and has a geodesic normal form. D_∞ is isomorphic to a subgroup of index 2 of itself, and is therefore not Hopfian.

A Coxeter group is a symmetry group generated by reflections. The group $W_{p,q,r}$ consists of the symmetries of the Euclidean plane generated by the reflections in the sides of a triangle with interior angles $\{\pi/p, \pi/q, \pi/r\}$. The 'two wrongs make a right' principle is a convenient way of reading a word left to right (opposite the convention from function composition) and changing the axes of reflection to obtain the same symmetry. Some other examples of Coxeter groups include D_n, D_∞, symmetries of the platonic solids and symmetries of the n-dimensional cube.

Chapter 3: Groups Acting on Trees

A group presentation consists of a set of generators and a set of defining relations; this defines a group. For example, $\mathbb{Z}^2$ is described by the presentation $\langle a, b \mid ab = ba \rangle$.

Consider n generators for a group, $S = \{x_1, \dots, x_n\}$. A word in the set $\{S \cup S^{-1}\}^*$ is freely reduced if it does not contain instances of an adjacent pair $x_i x_i^{-1}$ or $x_i^{-1} x_i$ as a subword. The free group on n generators, $\mathbb{F}_n$, has no freely reduced words representing the identity. The elements of $\mathbb{F}_n$ are equivalence classes of words, each with a freely reduced representative. We can therefore represent any element of $\mathbb{F}_n$ with a unique freely reduced word. The free group on n generators has the following presentation: $\mathbb{F}_n = \langle x_1, x_2, \dots, x_n \mid \quad \rangle$. It has no relations.

The Cayley graph for $\mathbb{F}_2$ is the uniformly 4-valent tree: an edge at each vertex for each generator and inverse. The Cayley graph of any free group must be a tree, as if it contained a cycle, that would correspond to a relation. The ping pong lemma is a way of identifying whether a group is free, typically subgroups of groups. For example, $SL_2(\mathbb{Z})$ and Homeo($\mathbb{R}$) both have subgroups generated by two elements which are free by the ping pong lemma. Surprisingly, $\mathbb{F}_2$ contains a finite index subgroup isomorphic to $\mathbb{F}_3$! Another useful characterization of free groups is that a group is free is and only if it acts freely on a tree. A key result which follows as a corollary is that every subgroup of a free group is free (but not necessarily finitely generated). An example of an infinitely generated subgroup of $\mathbb{F}_2$ is the commutator subgroup $[\mathbb{F}_2, \mathbb{F}_2] = \{[x^m, y^n] \mid m, n \in \mathbb{Z}/\{0\}\}$. This holds in $\mathbb{F}_n$.

Given a group G and a choice of group elements $g_1, \dots, g_n \in G$ (not necessarily nontrivial or distinct), there is a group homomorphism $\phi : \mathbb{F}_n \to G$ where $\phi(x_i) = g_i$ for all basis elements x_i of $\mathbb{F}_n$. By the first isomorphism theorem, it follows that every finitely generated group is a quotient of $\mathbb{F}_n$ for some n.

A free product of two groups, denoted $G * H$, is a way of combining two groups such that there is no commuting between the respective groups (unlike a direct product). In a sense, it is free 'between the groups'. The generating set of the free product $G * H$ is the disjoint union of the generating sets for G and H, and the set of defining relations is the disjoint union of the relations for G and H. Every free product of groups can be realized as a symmetry group of a biregular tree (a tree where all vertices have one of two valences, and these valences alternate between adjacent vertices).

A group is said to have some property *virtually* if it contains a finite-index subgroup which has that property. For example, D_∞ is not abelian, but it is virtually abelian, since it has $\mathbb{Z}$ as a finite index subgroup.

Serre's property Fixe Arbre (FA) relates to group actions on trees: a group has property FA if every action of G on a tree admits a fixed point. All finite groups have this property. An example of an infinite group which is FA is the automorphism group of the free group (for $n \geq 3$). A free group is not FA since its Cayley graph is a tree.

Chapter 4: Baumslag Solitar groups

A group action on a topological space $G \curvearrowright X$ is discrete if the orbits are discrete, for example $\mathbb{Z} \curvearrowright \mathbb{R}$. The action $G \curvearrowright X$ is proper if only finitely many group elements fix a given point in the space.

The Baumslag Solitar groups are two-generator, one-relator groups with the following presentation:

$$BS(m, n) = \langle a, b \mid ab^m = b^n a \rangle.$$

$BS(1, 2)$ can be viewed as an action on $\mathbb{R}$ as follows: take the subgroup of Homeo($\mathbb{R}$) generated by linear functions $a(x) = 2x$ and $b(x) = x + 1$. The action of $BS(1, 2)$ on $\mathbb{R}$ is neither discrete nor proper. The stabilizer of $0 \in \mathbb{R}$ is the subgroup generated by $a(x)$ which is isomorphic to $\mathbb{Z}$ and is thus infinite. The action is not discrete as $a^{-n} b(0) = \frac{1}{2^n}$, so the orbit of 0 is not discrete.

The Baumslag Solitar groups often come up as counterexamples due to their interesting properties. Certain BS groups are not Hopfian, for example $BS(2, 3)$. A residually finite group has the property that for any nontrivial element $g \in G$, there is a finite index normal subgroup which does not contain g. Not all of the Baumslag Solitar groups are residually finite, for example $BS(3, 4)$. In particular, $BS(m, n)$ is residually finite if and only if $m = 1$ or $|n| = m > 1$ [1]. A group is linear if it embeds into a matrix group. If a group is not residually finite, it cannot be linear, so $BS(2, 3)$ is not linear. $BS(1, 2)$ is.

Chapter 5: Words and Dehn's Word Problem

Dehn's word problem (1912) is as follows: given a group G with finite generating set S, can you determine whether an arbitrary word represents the identity element in G? Note that this is equivalent to asking if two distinct words represent the same element. The word problem is solvable in G if given any word, there exists an algorithm to decide if the word represents e in G (in finite time). The word problem in (G, S) is solvable if and only if you can determine which words correspond to closed paths in the Cayley graph. There are finitely presented groups with unsolvable word problem!

Given a group and generating set S, a normal form is a prescribed way of writing a group element using elements of $S \cup S^{-1}$. A normal form makes clear which words represent the same group element: just look at their normal forms. In $\mathbb{Z} \oplus \mathbb{Z}$, some normal forms consist of first moving vertically and then horizontally, or vice versa. 'Combing' refers to the effect a normal form has on a Cayley graph. Groups with normal forms along with an algorithm for putting an arbitrary word into normal form have solvable word problem.

Related questions about group presentations are the conjugacy problem and the isomorphism problem. The conjugacy problem asks, given (G, S), whether two arbitrary words in $\{S \cup S^{-1}\}^*$ are conjugates. The isomorphism problem asks if two group presentations (G, S) and (H, T) generate isomorphic groups. We don't even have an algorithm that decides whether a group presentation is trivial.

If a finitely presented group has solvable word problem, then all other presentations also have solvable word problem (this is seen using Tietze transformations) [6, chapter 2.1].

Chapter 6: A finitely generated, infinite torsion group

In 1902, Burnside asked if every finitely generated infinite group contains an element of infinite order (the Weak Burnside Problem). Surprisingly, the answer is no, and this chapter outlines an example.

You can easily construct infinite groups with all elements having finite order, for example the infinite product $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots$, or $\mathbb{Q}/\mathbb{Z}$, but these groups are *not finitely generated*. Our finitely generated infinite torsion group will be a subgroup of the symmetry group of a rooted, infinite ternary tree $\mathcal{T}$.

Consider the symmetry σ of $\mathcal{T}$ that permutes the three main branches of the tree according to the cycle (123). A version of this symmetry can also be applied by treating any vertex as the root, permuting any three branches of $\mathcal{T}$. The waterfall symmetry ω is an infinite product of σ's applied at vertices moving down and to the right. These two elements σ and ω generate a subgroup of the symmetry group of $\mathcal{T}$ which is infinite with all elements having order a power of 3.

The Burnside Problem asks if a finitely generated group has an upper bound n on the order of its elements, must it be finite? The answer is yes if the upper bound $n = 2, 3, 4$ or 6, and the groups are very large. For other small values of n, very little is known. It is open whether the group on two generators with every element satisfying $g^5 = e$ is finite. However, all groups on two or more generators with odd $n \geq 665$ are infinite.

Chapter 7: Regular languages and normal forms

Any subset of words in the free monoid of some alphabet $\{x_1, \ldots, x_n\}$ is a language. An automaton is a directed graph with its edges decorated according to an alphabet, and with certain vertices as start states and accept states. The language accepted by a given automaton is the set of all words that correspond to edge paths from a start state to an accept state.

A deterministic automaton is a finite-state automaton (finite graph) which has exactly one start state and no two edges leaving a vertex have the same label. A regular language is any language that is accepted by a deterministic automaton. The intersection, union, and concatenation of regular languages is regular.

A heuristic for determining whether a language is regular is if one would need unlimited memory in order to determine if an arbitrary word is in the language. The freely reduced words corresponding to the elements of $\mathbb{F}_2$ form a regular language. A language that is not regular is the set of words $\{a^n b^n \mid n \geq 1\}$.

In relation to the word problem of a group, the set of all words in $\{S \cup S^{-1}\}^*$ representing the identity element is a regular language if and only if the group is finite. The Muller-Schupp theorem proved in [7] states that G has context-free word problem if and only if G is virtually free.

A normal form is regular if its words form a regular language. If a regular normal form exists for some group G with finite generating set S, then any other finite generating set T for G also admits a regular normal form. If G and H have normal forms then so do $G \oplus H$ and $G * H$.

A result obtained via regular languages is that the intersection of finitely generated subgroups of a free group is a finitely generated free group (Howson's Theorem). This is an improvement to the result from chapter 3 about subgroups of free groups being free—we have finite generation.

Chapter 8: The lamplighter group

A semi-direct product $H \rtimes G$ is like a direct sum in that it is formed by the multiplication of two groups, but elements do not commute freely between the groups. Commuting an element of the first subgroup across the second comes at a cost, determined by a homomorphism $\phi : G \to \mathrm{Aut}(H)$. If we write $\phi(g)(h) =: g \cdot h$ then we can define multiplication of elements by

$$[h_1, g_1] * [h_2, g_2] = [h_1(g_1 \cdot h_2), g_1 g_2].$$

The operation $*$ then defines a group structure on $H \times G$, denoted $H \rtimes_\phi G$ [3, chapter 5.5].

The dihedral group D_n is an example of a semi-direct product: $D_n \simeq \mathbb{Z}_n \rtimes \mathbb{Z}_2$. Similarly, the infinite dihedral group $D_\infty \simeq \mathbb{Z} \rtimes \mathbb{Z}_2$.

A wreath product $G \wr H$ is a special type of semi-direct product formed using a (possibly infinite) direct sum. The lamplighter groups are examples of wreath products. Imagine evenly spaced lampposts along an infinite street, and a lamplighter responsible for lighting some of the bulbs. Each element of the lamplighter group L_2 corresponds to a finite set of illuminated bulbs along with a final position of the lamplighter [2, chapter 15]. L_2 is generated by two elements, which may be surprising as it contains the infinite product $\cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$ as a subgroup. However, L_2 is *not* finitely presented.

Chapter 9: The Geometry of Infinite groups

Given a group G with a finite generating set, the word length of a group element g is the minimal length of a word representing g. The distance between two elements of G is the length of the word $g^{-1}h$. This defines a metric on G called the *word metric*.

Gromov's corollary states that every finitely generated group has a faithful representation as a group of isometries of a metric space, namely G with the word metric. A word ω representing g is a geodesic if it has minimal length. Minimal length paths in the Cayley graph are called geodesic paths.

For a finite group G, the diameter of its Cayley graph is the smallest integer which is an upper bound on the distance between two arbitrary elements in G. The symmetry group of the Rubik's Cube has over 43 quintillion elements, but it has diameter ≤ 26.

For a group and finite generating set, the *growth series* is a series with general term the size of the sphere of radius n times z^n for all $n \geq 0$. The growth series of a direct sum $G \oplus H$ is the product of the growth series for G and growth series for H.

A growth series is rational if it is the power series of some rational function. $\mathbb{Z}$ with one generator has rational growth series. A growth series of a finitely generated group which is not rational is the lamplighter group where lamps are being lit on the Cayley graph of $\mathbb{F}_2$.

Chapter 11: Large scale geometry of groups

The Cayley graph is a useful construction for studying a group's structure. However, the Cayley graph is dependent on the generating set chosen. We want to focus on properties of the group that are independent of this choice. Changing generating sets can have drastic effects on the *local* structure of a Cayley graph, so we consider large-scale, or geometric, properties of G—namely the properties which are invariant under changes in generating set.

Given two generating sets S, T for G, there is a constant $\lambda \geq 1$ such that for any $g, h \in G$, $\frac{1}{\lambda} d_S(g,h) \leq d_T(g,h) \leq \lambda d_S(g,h)$ (where d_* is the word metric in the respective generating set). For example, the Cayley graph of $\mathbb{Z}$ with generating set $S = \{1\}$ is a line, whereas the Cayley graph with generating set $T = \{2, 3\}$ is more of a tangled braid-like thing. However, when you zoom out, both look roughly like a line.

The growth function of a group is the size of the ball of radius n about the identity, $f(n) = |B(n, e)|$. The growth of a group is the rate at which the growth function grows as a function of n. This rate—whether linear, polynomial, exponential, etc.—is independent of the choice of generating set, and is thus a geometric property of G. The group $\mathbb{Z}^N = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ has polynomial growth of degree N, and the free group on two or more generators has growth at least exponential. All finitely generated groups have growth rate dominated by 2^n. A key result from Gromov in 1981 is that a finitely generated group has polynomial growth if and only if it is virtually nilpotent [5].

The upper central series of a group G is a chain of subgroups defined inductively: the first is the trivial subgroup, the next is the center of G, and each subgroup is normal in the next, i.e. $1 = Z_0 \lhd Z_1 \lhd \dots$, such that the quotient Z_n/Z_{n-1} is the center of G/Z_{n-1}. A *nilpotent* group G is one whose upper central series terminates with G.

Milnor asked the following question: are there groups which have subexponential growth, i.e., growth strictly between n^d and 2^n? The answer is yes, proved by Grigorchuck in 1983. An example is similar to the infinite torsion group from chapter 6; for further reading, see [4].

The ends of a group are the connected, unbounded components in the complement of the ball of radius n, as $n \to \infty$. The number of ends a group has is a geometric property. A group has no ends if and only if it is finite, as for some n, $B(n, e)$ contains the entire group. The Freudenthal-Hopf Theorem states that every finitely generated group has either zero, one, two, or infinitely many ends. Some examples are $\mathbb{Z} \oplus \mathbb{Z}$ with one end, $\mathbb{Z}$ with two ends, and $\mathbb{F}_2$ with infinitely many ends. Loosely, two metric spaces are quasi-isometric if they are reasonably close to being isometric, i.e. if there's a map between them which allows for bounded multiplicative and additive errors [2, chapter 7]. Metric spaces associated to different generating sets for G are quasi-isometric, so we care about properties of G which are invariant under quasi-isometry. Given a finitely generated group G and two finite generating sets S and T, G with the word metric induced by S is quasi-isometric to G with the word metric from T (the quasi-isometry type is a property of the underlying group, not specific to a generating set).

A finite-index subgroup H of some group G is finitely generated if and only if G is finitely generated. Theorem 11.46 states that if G and H are finitely generated groups that are quasi-isometric, they have the same growth rates and the same number of ends.

References

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