SAMPLE PURE MATH PRELIMINARY EXAM

A. Core Material

Answer four of the following six questions.

Problem A.1. Consider the matrix $A = \begin{bmatrix} -28 & 18 \\ -54 & 35 \end{bmatrix}$.

(a) Find a real matrix B such that $B^3 = A$.
(b) How many distinct complex matrices X satisfy $X^3 = A$?

Problem A.2. Let n be a positive integer and V_n the complex vector space of $n \times n$ complex matrices. For A, $B \in V_n$, define $\langle A, B\rangle = \operatorname{tr}(AB^*)$, where B^* denotes the conjugate transpose of B and $\operatorname{tr}(X)$ denotes the trace of a matrix X.

(a) Show that $\langle \cdot, \cdot \rangle : V_n \times V_n \to \mathbb{C}$ as above defines an inner product on V_n.
(b) Find the orthogonal complement of the subspace of diagonal matrices.

Problem A.3. Suppose $\{x_n\}$ is a sequence in a complete metric space (X, d) such that

$$\sum_n d(x_n, x_{n+1})$$

is a convergent series. Show that the sequence $\{x_n\}$ is convergent in X.

Problem A.4. (a) Show that

$$\int_0^\infty \frac{\cos x}{x^{1/3}}\, dx$$

converges (as an improper integral).
(b) Show that the integral in part (a) does not converge absolutely.

Problem A.5. Find all the one-to-one conformal mappings φ from the unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$ to the upper-half plane $H^+ = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ satisfying $\varphi(1/2) = i$.

Problem A.6. (a) Suppose f is analytic in a domain D, and $|f(z)| \leq 10$ for all $z \in D$. Show that

$$|f'(z)| \leq \frac{10}{d(z)},$$

where $d(z)$ is the distance from z to the boundary of D. (Hint: use the Cauchy Integral Formula.)
(b) State and prove Liouville's theorem for bounded entire functions.

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B. Probability and Statistics Topics

Answer three of the following four questions.

Problem B.1. A balanced coin is tossed 10 times. What is the probability of obtaining "heads" at least 4 times in a row?

Problem B.2. Suppose that X and Y are continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Give reasons for your answer.
(b) Let $Z = X/Y$. Show that Y and Z are independent random variables and find the marginal density functions of Y and Z.

Problem B.3. Let $X_1, X_2, ..., X_n$ be a random sample from the Uniform$[0, \theta]$ distribution.

(a) Find an estimator $\widetilde{\theta}$ of θ by the method of moments.
(b) Find an estimator $\widehat{\theta}$ of θ by the method of maximum likelihood.
(c) Compare $\widetilde{\theta}$ and $\widehat{\theta}$ on the basis of bias and variance.

Problem B.4. A breeder of golden retriever dogs wants to be able to predict the size of the litter from the age of the bitch, and has kept records for several years.

Bitch:	1	2	3	4	5	6	7	8	9	10	11	12	13
Age (yr):	2.0	2.5	4.0	3.5	6.0	5.0	4.5	4.0	8.0	2.5	3.0	3.5	4.0
Litter size:	11	10	9	12	5	9	9	8	7	10	12	10	9

Present an analysis in an ANOVA table with F-Tests for non-linearity and for the slope of the regression line. Give a 95% confidence interval for the residual variance. State your assumptions and your conclusions. Where possible, test your assumptions. Could you use this analysis to predict the age at which a bitch will become infertile?