APPLIED MATH PRELIMINARY EXAM

Please answer four questions on part A and three questions on part B. All questions are weighted evenly. Please provide clear and complete explanations of all steps taken, and make sure to justify any assumptions you make in the process. Good luck!

A. Core material

Answer four of the following six questions.

Problem A.1. Consider the matrix $A \in M_{2\times 2}(\mathbb{R})$ given by $A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}$.

(a) Show that A is diagonalizable over $\mathbb{Z}$.
(b) Find all matrices $X \in M_{2\times 2}(\mathbb{R})$ that satisfy $X^2 = A$. How many distinct square roots does A have?

Problem A.2. For $x, y \in \mathbb{R}$, let $M = M(x, y)$ be the 3×3 matrix

$$M(x,y) = \begin{bmatrix} x & y & y \\ x & x & y \\ x & x & x \end{bmatrix}.$$

Show that $\operatorname{rank}(M)$ takes on all possible values, and determine the sets of (x, y) such that

(a) $\operatorname{rank} M(x, y) = 3$.
(b) $\operatorname{rank} M(x, y) = 2$.
(c) $\operatorname{rank} M(x, y) = 1$.
(d) $\operatorname{rank} M(x, y) = 0$.

Problem A.3. (a) State the Weierstrass approximation theorem.
(b) Let $\varepsilon > 0$ and suppose $f : [0, 1] \to \mathbb{R}$ is a differentiable function whose derivative f' is continuous on $[0, 1]$ (i.e. f is C^1 on $[0, 1]$). Prove that there exists a polynomial p such that $|f(x) - p(x)| \le \varepsilon$ and $|f'(x) - p'(x)| \le \varepsilon$ for all $x \in [0, 1]$.

Problem A.4. Suppose f and f' are continuous on $[0, \infty]$ and $f(x) = 0$ for $x \ge 10^{10}$. Show that

$$\int_0^\infty f(x)^2 dx \le 2\sqrt{\int_0^\infty x^2 f(x)^2 dx}\sqrt{\int_0^\infty f'(x)^2 dx}.$$

Problem A.5. (a) Give a precise statement of the fundamental theorem of algebra.
(b) Prove the fundamental theorem of algebra.

Problem A.6. Let a be a complex number with $|a| > 1$. Evaluate the path integral around the unit circle in $\mathbb{C}$:

$$\int_{|z|=1} \frac{|dz|}{|az - 1|^2}.$$

(Note that $|dz|$ represents integration with respect to arc-length.)

Date: January 5, 2006.

B. Applied math

Answer three of the following four questions.

Problem B.1. Consider the system of three differential equations,

$$\begin{aligned}\dot{x} &= y - x\ ,\\ \dot{y} &= ax - y - xz\ ,\\ \dot{z} &= xy - z,\end{aligned}$$

where $a > 0$ is parameter.

(i) Find all critical points of the system..
(ii) For each critical point, construct the matrix of linearization, compute its eigenvalues, and classify stability of the critical point.
(iii) Describe what happens when a passes through the value $a = 1$.

Problem B.2. Consider the system of two differential equations

$$\begin{aligned}\dot{x} &= 2x + 2y - x(2x^2 + y^2)\ ,\\ \dot{y} &= -2x + 2y - y(2x^2 + y^2).\end{aligned}$$

(i) Rewrite the system of equations in polar coordinates (r, θ)
(ii) Describe all critical points of the problem and their stability
(iii) Use the Poincare-Bendixson theorem to show that at least one limit cycle solution exists on the phase plane. Estimate the location of the limit cycle.

Problem B.3. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad x \in [0, \pi],$$

subject to the boundary conditions

$$\phi(0) = 0, \qquad \phi(\pi) - 2\phi'(0) = 0.$$

(i) Determine all positive eigenvalues.
(ii) Determine all negative eigenvalues.
(iii) Is $\lambda = 0$ an eigenvalue? Do there exist other eigenvalues? If so, where are they located? Explain the answers.

Problem B.4. Consider the wave equation with friction and time-periodic forcing term,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t} + g(x)\cos\omega t,$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0$$

and the initial conditions

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = 0,$$

where β is small $(0 \leq \beta \leq 2\pi)$ and $f(x)$,$g(x)$ are smooth functions on $x \in [0, 1]$ satisfying the Dirichlet boundary conditions.

(i) Find a general solution of the problem by either separation of variables or by Fourier series.
(ii) Write down the closed form solution when $f(x) = 0$ and $g(x) = \sin(\pi x)$.
(iii) Use the closed form solution and find the values of ω when the solution $u(x, t)$ grows unbounded as $t \to \infty$ in the case $\beta = 0$. Confirm the unbounded growth of $u(x, t)$.